Set Theory

Winter 2014/2015 – Dr. P. Lücke, Dr. P. Schlicht – Bonn University
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Some introductory literature is Schimmerling[9], Devlin[2]. Results from mathematical logic are not used, but can be found in any textbook on mathematical logic. These notes are typed by Timo Weiß.

Note that the second part of the lecture beginning with section five is not proofread. Exam date: Monday, 9th February 2015 (end of lecture period: Fri, 6th February 2015)

Contents

1 The Framework of Set Theory **3**
1.1 The Language of Set Theory . . . 3
1.2 Russell's Paradox . . . 4
1.3 The Set Existence Axiom . . . 4
1.4 The Extensionality Axiom . . . 5
1.5 Classes . . . 5
1.6 The Pairing Axiom . . . 6
1.7 Relations and Functions . . . 7
1.8 The Union Axiom . . . 8
1.9 The Infinity Axiom . . . 9
1.10 The Foundation Axiom . . . 9
1.11 The Separation Scheme . . . 9
1.12 The Replacement Scheme . . . 10
1.13 The Axiom System of Zermelo-Fraenkel without the Power Set Axiom . . . 10
1.14 Induction and Recursion . . . 11
1.15 Ordinals . . . 12
1.16 The Mostowski Collapse . . . 16
1.17 Ordinal Arithmetic . . . 17
1.18 The Von Neumann Hierarchy . . . 18
1.19 The Real Numbers . . . 19

2 Cardinals **21**
2.1 Cardinals and the Axiom of Choice . . . 21
2.2 Cardinal Arithmetic . . . 26
2.3 Infinite Sums and Products . . . 30
2.4 Cofinality . . . 32
2.5 Cardinal Exponentiation . . . 35

3 Applications of the Axiom of Choice **40**
3.1 Various Applications 40
3.2 Filters and Ultrafilters 45

4 Club Sets and Stationary Sets **46**
4.1 The Club Filter 46
4.2 Splitting Stationary Sets 48
4.3 Silver's Theorem 51

5 Trees and Partitions **54**
5.1 Introduction 54
5.2 König's Lemma 59
5.3 Ramsey's Theorem 60
5.4 The Suslin Problem 65
5.5 Aronszajn Trees and Walks on Ordinals 71
5.6 Partitions and Weakly Compact Cardinals 79
5.7 Special Aronszajn Trees and Mahlo Cardinals 83

6 Diamonds **92**
6.1 Diamond Principles and Suslin Trees 92
6.2 Diamond Principles and the Generalised Continuum Hypothesis 96

References **99**

Index **100**

List of Lectures

Lecture 1 ...06.Oct.....3
Lecture 2 ...08.Oct.....8
Lecture 3 ...13.Oct....12
Lecture 4 ...15.Oct....16
Lecture 5 ...20.Oct....21
Lecture 6 ...22.Oct....24
Lecture 7 ...27.Oct....27
Lecture 8 ...29.Oct....30
Lecture 9 ...05.Nov....32
Lecture 10 ...10.Nov....36
Lecture 11 ...12.Nov....39
Lecture 12 ...17.Nov....41
Lecture 13 ...19.Nov....45
Lecture 14 ...24.Nov....48
Lecture 15 ...26.Nov....51
Lecture 16 ...01.Dec....54
Lecture 17 ...08.Dec....57
Lecture 18 ...10.Dec....59
Lecture 19 ...15.Dec....61
Lecture 20 ...17.Dec....64
Lecture 21 ...22.Dec....67
Lecture 22 ...07.Jan....71
Lecture 23 ...12.Jan....74
Lecture 24 ...14.Jan....77
Lecture 25 ...19.Jan....80
Lecture 26 ...21.Jan....83
Lecture 27 ...26.Jan....85
Lecture 28 ...28.Jan....88
Lecture 29 ...02.Feb....91
Lecture 30 ...04.Feb....94

1 The Framework of Set Theory

In this course, we develop the theory of sets, built up from the empty set. We will consider themes such as Lecture 1 6th Oct

- How do transfinite limit processes work?
- How can arithmetic be generalized to the infinite?
- What can we say about the sizes of infinite sets, for example the set of real numbers?
- Infinite combinatorics: What kinds of uncountable trees and other uncountable structures exist?

Results in infinite combinatorics have consequences in many fields of mathematics. All (or most) mathematical objects, for example real functions, groups, fields, topological spaces, Banach spaces, ultrafilters, can be formalized as sets, therefore set theory is a framework for mathematics. The formalization is straightforward (but sometimes tedious).

Example 1.0.1. 1. A *natural number* is of the form $0 := \emptyset$ or $n+1 := n \cup \{n\}$ for some natural number n.

2. An *ordered pair* (x, y) is the set $\big\{\{x\}, \{x, y\}\big\}$.
3. A *rational number* is an equivalence class of an *ordered triple* of natural numbers, where (m, n, k) represents $\frac{m-n}{k+1}$.
4. A *real number* r is the left half of a Dedekind cut in $\mathbb{Q}$, i.e. the set $L = \{q \in \mathbb{Q} | q \leq r\}$.
5. A *function* $f : \mathbb{R} \to \mathbb{R}$ is the set of pairs $(x, f(x))$ for $x \in \mathbb{R}$.

1.1 The Language of Set Theory

In the beginning, we will try to be as precise and formal as possible. In other fields of mathematics, this is usually not necessary in this form. However in set theory, when considering very large sets and collections of sets, it is easy to reach a contradiction, if one is not careful about the rules for the formation of sets, as in Russell's paradox. Therefore we try to be as precise as possible to ensure that every statement can be translated into a formal statement in the language of set theory.

Definition 1.1.1. 1. The *formal language of set theory* is $L = \{\in\}$.

2. A *set theoretic formula* or $\in$-*formula* is a first order statement in the language of set theory using the logical symbols $\wedge, \vee, \neg, \rightarrow, \exists, \forall, =, (,)$, variables and the relation symbol $\in$, for example "$\forall x \exists y\, x \in y$".

We will not actually work in this formal language, but introduce many definitions and abbreviations, in order to write statements such as $\sup_{x<y} f(x) = f(y)$ or $e^{ix} = -1$.

Another reason to ensure that every result is proved from certain axioms is that research in set theory is mainly concerned with models of set theory, i.e. models of a formal theory, just as group theory is concerned with the formal theory of groups, and for this reason the formalization is necessary.

1.2 Russell's Paradox

In the early days of set theory, people tried to describe informally what is a set. Georg Cantor characterises sets as follows:

> Unter einer ‚Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unserer Anschauung oder unseres Denkens (welche die ‚Elemente' von M genannt werden) zu einem Ganzen.[1, p. 481]

This roughly translates to:

> By the notion of a 'set' we mean any collection M of certain distinct objects m of our experience or intellect (which are called the 'elements' of M) into a whole.

Felix Hausdorff characterises sets as follows:

> Eine Menge ist eine Zusammenfassung von Dingen zu einem Ganzen, d.h. zu einem neuen Ding.[4, p. 1]

This roughly translates to:

> A set is a collection of things into a whole, i.e. a new thing.

If we are not careful, these definitions of sets lead to the following contradiction.

Russell's paradox. Suppose that there is a set x which consists of exactly the sets y with $y \notin y$. If $x \in x$, then $x \notin x$, but if $x \notin x$, then $x \in x$, contradiction.

To avoid this problem, it is necessary to formulate axioms for the formation of sets, which we introduce in the following. Once an axiom or axiom scheme is introduced, it is used without comment in the following proofs.

1.3 The Set Existence Axiom

The Set Existence Axiom states the existence of a set, in particular of an empty set.

Axiom (Set Existence)**.**

$$\exists x \forall y\, y \notin x.$$

1.4 The Extensionality Axiom

The Extensionality axiom states that every set is determined by its elements.

Axiom (Extensionality)**.**

$$\forall x \forall x' (\forall y (y \in x \leftrightarrow y \in x') \rightarrow x = x').$$

Lemma 1.4.1. $\forall x \forall x' ((\forall y\, y \notin x \wedge \forall y\, y \notin x') \rightarrow x = x')$ – *The empty set is uniquely determined.*

Proof. If $\forall y\, y \notin x, \forall y\, y \notin x'$, thenn x and x' have the same elements. So, $x = x'$, by Extensionality. □

1.5 Classes

We need a notation for the universe of sets which is itself not a set. It is formalised as a class.

Definition 1.5.1. A *class* or *class term* $A = \{x | \varphi(x, s_0, \ldots, s_n)\}$ is given by a first-order formula φ and sets $s_0, \ldots, s_n$.

Definition 1.5.2. Suppose that $A = \{x | \varphi(x, s_0, \ldots, s_n)\}$ is a class and s is a set.

1. $s \in A$ if $\varphi(s, s_0, \ldots, s_n)$ holds.
2. Let $s = A$ if $\forall x \in s\, x \in A \wedge \forall x \in A\, x \in s$.

Definition 1.5.3. Suppose that A, B are classes.

1. $A \subseteq B$ if $\forall x \in A\, x \in B$.
2. $A = B$ if $A \subseteq B$ and $B \subseteq A$.

Lemma 1.5.4. *If A, B are classes and s, t are sets with $A = s$ and $B = t$, then $s = t$ if and only if $A = B$.*

Proof. By the Extensionality Axiom. □

We will identify a class A with a set s if $A = s$.

Definition 1.5.5.

1. $\emptyset := \{x | x \neq x\}$ "the empty class/set".
2. $V := \{x | x = x\}$ "the universe of sets".
3. $\{x_0, \ldots, x_n\} := \{x | x = x_0 \vee \ldots \vee x = x_n\}$.

Lemma 1.5.6. $\emptyset \in V$. *The empty class is a set.*

Proof. By the Set Existence Axiom there is a set $s = \emptyset$. So $\emptyset = s \in V$. □

Definition 1.5.7. A class A is called a *proper class* if there is no set s with $A = s$.

Definition 1.5.8. Suppose that $A, B, A_0, \ldots, A_n$ are classes.

1. $A_0 \cup \ldots \cup A_n := \{x | x \in A_0 \vee \ldots \vee x \in A_n\}$.
2. $A_0 \cap \ldots \cap A_n := \{x | x \in A_0 \wedge \ldots \wedge x \in A_n\}$.
3. $A \setminus B := \{x | x \in A \wedge x \notin B\}$.
4. $\bigcup A = \bigcup_{x \in A} x := \{y | \exists x \in A\, y \in x\}$.
5. $\bigcap A = \bigcap_{x \in A} x := \{y | \forall x \in A\, y \in x\}$.

Lemma 1.5.9. $\bigcup\{x, y\} = x \cup y$.

Proof. "$\subseteq$": Suppose $u \in \bigcup\{x, y\}$. Then there is some $v \in \{x, y\}$ with $x \in v$. We can assume that $v = x$, so $u \in x$. So $u \in x \cup y$.

"$\supseteq$": Suppose that $u \in x \cup y$. Suppose that $u \in x$. So, $u \in \bigcup\{x, y\}$.

□

Every statement about classes can be translated into a statement in the language of set theory. Note that we don't quantify over classes, i.e. we do not consider statements of the form 'there is a class A with certain property', since we cannot quantify over formulas, only over sets.

1.6 The Pairing Axiom

The Pairing axiom states that for any sets s, t, there is a set which has exactly the elements s, t, i.e. $\{s, t\} \in V$.

Axiom (Pairing)**.**

$$\forall x \forall y \exists z\, \forall u (u \in z \leftrightarrow (u = x \vee u = y)).$$

Definition 1.6.1. Suppose that $s, t, s_0, \ldots, s_{n+1}$ are sets.

1. $(s, t) := \big\{\{s\}, \{s, t\}\big\}$ *ordered pair*.
2. $(s_0, \ldots, s_{n+1}) := ((s_0, \ldots, s_n), s_{n+1})$ *(ordered) tuple*.

Lemma 1.6.2. *1.* $\forall x \forall y \exists z\, z = (x, y)$.

2. $\forall x_0 \ldots \forall x_n \exists z\, z = (x_0, \ldots, x_n)$.

Proof. 1. Suppose that s, t are sets. By the Pairing Axiom, there are sets u, v with $u = \{x\}$ and $v = \{x, y\}$. Again, by the Pairing Axiom, there is a set z with $z = \{u, v\} = (s, t)$.

2. By induction on n.

□

The definition of the ordered pair satisfies the *fundamental propoerty of ordered pairs*:

Lemma 1.6.3.

$$\forall x, y, x', y' \big((x, y) = (x', y') \to (x = x' \wedge y = y')\big).$$

Proof. Suppose that $(x, y) = (x', y')$.

Case 1. $x = y$. *Then* $\{x\} = \{x, y\}$ *and* $(x, y) = \{\{x\}\}$. *Then* $\{\{x'\}, \{x', y'\}\} = (x', y') = (x, y) = \{\{x\}\}$. *Then* $x' = y', x = x', y = y'$.

Case 2. $x \neq y$. *Then, by Extensionality,* $x' = x$ *or* $x' = x = y$. *Since the latter case leads to a contradiction,* $x' = x$. *Hence* $\{x, y\} = \{x', y'\}$ *and since* $y \neq x = x', y = y'$.

□

Definition 1.6.4. Suppose that $A_0, \dots, A_n$ are classes.
Let $A_0 \times \dots \times A_n := \{(x_0, \dots, x_n) | x_0 \in A_0 \wedge \dots \wedge x_n \in A_n\}$.

1.7 Relations and Functions

The notion of an ordered pair allows us to define relations and functions.

Definition 1.7.1.

1. A class R is a (binary) *relation* (on a class A) if its elements are indeed pairs in $A \times A$. We also write xRy for $(x, y) \in R$.
2. A relation F is a *function* or *map* if $\forall x, y, y' (xFy \wedge xFy' \to y = y')$. We also write $F(x) = y$ for xFy.

Definition 1.7.2. Suppose that R, S, A are classes.

1. $\mathrm{dom}(R) := \{x | \exists y\, (x, y) \in R\}$. "domain"
2. $\mathrm{range}(R) := \{x | \exists y\, (y, x) \in R\}$. "range"
3. $\mathrm{field}(R) := \mathrm{dom}(R) \cup \mathrm{range}(R)$. "field"
4. $R{\upharpoonright}A := \{(x, y) | xRy \wedge x \in A\}$ "restriction"
5. $R[A] = R''A := \{y | \exists x \in A\, (x, y) \in R\}$. "image"
6. $R^{-1}[A] := \{x | \exists y \in A\, (x, y) \in R\}$. "preimage"
7. $S \circ R := \{(x, z) | \exists y\, (x, y) \in R \wedge (y, z) \in S\}$. "composition"
8. $R^{-1} := \{(y, x) | (x, y) \in R\}$.

Definition 1.7.3. Suppose that R is a relation.

1. R is called *reflexive* if $\forall x \in \mathrm{field}(R)\, (x, x) \in R$.
2. R is called *irreflexive* if $\forall x \in \mathrm{field}(R)\, (x, x) \notin R$.
3. R is called *symmetric* if $\forall x, y \in \mathrm{field}(R)\, \big((x, y) \in R \leftrightarrow (y, x) \in R\big)$.

4. R is called *antisymmetric* if $\forall x, y \in \mathrm{field}(R)\ \big((x,y) \in R \wedge (y,x) \in R \to x = y\big)$.
5. R is called *connex* or *linear* if $\forall x, y \in \mathrm{field}(R)\ (x = y \vee xRy \vee yRx)$.
6. R is called *transitive* if $\forall x, y, z \in \mathrm{field}(R)\ (xRy \wedge yRz \to xRz)$.
7. R is an *equivalence relation* if R is reflexive, symmetric and transitive.
8. If R is an equivalence relation and $x \in \mathrm{field}(R)$, then $[x]_R := \{y \in \mathrm{field}(R) | xRy\}$ the *equivalence class* of x.

Definition 1.7.4. Suppose that R is a relation, A is a class.

1. R is a *partial order* if R is reflexive, transitive and antisymmetric. We often denote partial orders by the symbol '$\leq$'.
2. R is a *linear order* if R is a partial order and R is linear.
3. R is a partial order on A if R is a partial order with $\mathrm{field}(R) = A$.
4. R is a *strict partial order* if it R is irreflexive, transitive and antisymmetric. We often denote strict orders by the symbol '$<$'.
5. R is a *strict linear order* if R is a strict partial order and R is linear.

Definition 1.7.5. Suppose that F is a function, A, B are classes.

1. F is a *function from A to B* $(F : A \to B)$ if $\mathrm{dom}(F) = A$ and $\mathrm{range}(F) \subseteq B$.
2. F is called a *partial function* from A to B $(F : A \rightharpoonup B)$ if $\mathrm{dom}(F) \subseteq A$ and $\mathrm{range}(F) \subseteq B$.
3. $F : A \to B$ is called *surjective* or *onto* if $\mathrm{range}(F) = B$.
4. $F : A \to B$ is called *injective* or *one-to-one* if $\forall x, x' \in A\ (x \neq x' \to F(x) \neq F(x'))$.
5. ${}^A B := \{f | f : A \to B\}$.

1.8 The Union Axiom

The Union Axiom states that the union $\bigcup s$ of a set s is again a set. Lecture 2 8th Oct

Axiom (Union).

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists u (u \in x \wedge z \in u)).$$

Lemma 1.8.1. $\forall x_0 \ldots \forall x_n\ \{x_0, \ldots, x_n\} \in V$.

Proof. By Pairing, this holds for $n = 0, 1$. Suppose that this holds for some $n \geq 1$. Then $\{x_0, \ldots, x_{n+1}\} = \bigcup \big\{\{x_0, \ldots, x_n\}, \{x_{n+1}\}\big\}$ is a set by Pairing and Union. $\square$

1.9 The Infinity Axiom

The Infinity Axiom states that there is an infinite set.

Definition 1.9.1.
1. For s a set, let $s + 1 := s \cup \{s\}$.
2. A set s is called *inductive* if $\emptyset \in s$ and $\forall x(x \in s \to x + 1 \in s)$.

More precisely, the Infinity Axiom states the existence of an inductive set.

Axiom (Infinity).
$$\exists y(\emptyset \in y \wedge \forall x(x \in y \to x + 1 \in y)).$$

1.10 The Foundation Axiom

The Foundation Axiom states that every set has an $\in$-minimal element.

Axiom (Foundation).
$$\forall y \exists x(x \in y \wedge x \cap y = \emptyset).$$

Lemma 1.10.1. *There are no $\in$-cycles, i.e. there are no sets $x_0, \ldots, x_n$ with $x_0 \in x_1 \in \ldots \in x_n \in x_0$.*

Proof. Suppose there is an $\in$-chain $x_0, \ldots, x_n$ as described above. Let $y = \{x_0, \ldots, x_n\}$. By Foundation, y has an $\in$-minimal element x_k. If $k = 0$, $x_n \in x_k \cap y$ which contradicts the Foundation Axiom. If $1 \leq k \leq n$, $x_{k-1} \in x_k \cap y$ which also contradicts the Foundation Axiom. Hence, there is no $\in$-chain. $\square$

The Foundation Axiom implies that the universe of sets is built up in a cumulative hierarchy.

The Foundation axiom is no restriction in the following sense. Given a model of the remaining axioms and schemes, one can form the class of well-founded sets x, i.e. such that x is an element of a transitive set y such that $(y, \in)$ is well-founded. One can show that this class is a model of all axioms and schemes including the Foundation Axiom.

1.11 The Separation Scheme

We consider schemes which generate infinitely many formulas.
The Separation Scheme states that any subclass of a set is again a set.

Axiom Scheme (Separation).
$$\forall x \, \forall x_0 \ldots \forall x_n \, \exists y \, \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(z, x_0, \ldots, x_n)).$$

Let A be the class definied by φ. $x \cap A$ is then a set.
Unrestricted Separation leads to Russell's paradox. This implies that the universe of sets is itself not a set.

Lemma 1.11.1. $V \notin V$.

Proof. Suppose that V is a set. Then $\{x|x \notin x\}$ is a set by Separation. By Russell's paradox, this is not a set, contradiction. □

Lemma 1.11.2. *There is a $\subseteq$-least inductive set.*

Proof. By Infinity, there is an inductive set x.
Then $y = \{u|u \in x \wedge \forall v(\mathsf{inductive}(v) \rightarrow u \in v)\}$ is a set by Separation. This is the intersection of all inductive sets. y is inductive, because $\emptyset \in y$ and if $w \in y$ and z is inductive, then we have $w \in z$, so $w+1 \in z$ by the inductivity of z. Hence, $w+1 \in y$. □

1.12 The Replacement Scheme

Axiom Scheme (Replacement)**.**

$$\text{If } F \text{ is a function, } \forall x\, F[x] \in V.$$

There are some redundancies within the axioms. The Seperation Scheme, for example, follows from the Replacement Scheme and the Set Existence Axiom.

Lemma 1.12.1. *All axioms without the Separation Scheme, but with the Replacement Scheme, already imply the Separation Scheme.*

Proof. Suppose x is a set and A is a class. If $A \cap x = \emptyset$, then $A \cap x$ is a set by the Set Existence Axiom. If $A \cap x \neq \emptyset$, then there exists an $u_0 \in A \cap x$. We define a function

$$F : x \to x, F(u) = \begin{cases} u, & \text{if } u \in A \\ u_0, & \text{if } u \notin A \end{cases}$$

Then $F[x] = A \cup x$ is a set by the Replacement Scheme. □

1.13 The Axiom System of Zermelo-Fraenkel without the Power Set Axiom

Definition 1.13.1. The axiom system ZF^- (Zermelo-Fraenkel set theory without the Power Set Axiom) consists of the axioms and schemes:

- Set Existence
- Extensionality
- Pairing
- Union
- Infinity
- Foundation
- Separation Scheme

- Replacement Scheme

Remark 1.13.2. There are some redundancies among the axioms.

- Set Existence is entailed by Infinity and Separation.
- Separation is entailed by Replacement and Set Existence.
- Pairing is also following from the other axioms (exercise).

Remark 1.13.3.
1. ZF^- is not finitely axiomatisable, in particular, the schemes cannot be replaced by axioms.
2. If ZF^- is consistent, ZF^- is incomplete, i.e. there are $\in$-sentences φ such that neither φ nor $\neg\varphi$ can be formally derived from ZF^- (Gödel Incompleteness Theorems).

1.14 Induction and Recursion

Definition 1.14.1. Suppose that $<$ is a relation.

1. If y is a set, let $\mathrm{pred}_<(x) = \{y|y < x\}$.
2. $<$ is *(strongly) well-founded* if for every set y,
   (i) $\mathrm{pred}_<(y)$ is a set ("strongly…", "set-like")
   (ii) if $y \cap \mathrm{field}(<) \neq \emptyset$, then there is an $x \in y$ which is $<$-minimal in y, meaning there is no $z \in y$ with $z < x$.
3. $<$ is a *well-order* if $<$ is well-founded and a linear order. $<$ is a well-order *on a class* A if $<$ is a well-order with $\mathrm{field}(<) = A$, or $<$ is the empty well-order and A has at most one element.

Theorem 1.14.2 (Induction for sets)**.** *Suppose that $<$ is a well-founded relation on a set u, $\varphi(x, y)$ is an $\in$-formula, and v is a set. If for all $y \in u$:*

$$(\forall x < y\, \varphi(x, v)) \to \varphi(y, v),$$

then $\varphi(x, v)$ holds for all $x \in u$.

Proof. Let $S := \{y \in u|\neg\varphi(y, v)\}$. This is a set by Separation. Suppose that $S \neq \emptyset$. Since $<$ is well-founded there is some $y \in S$ with $\mathrm{pred}_<(y) \cap S \neq \emptyset$. Then $\varphi(y, v)$ holds, contradiction. □

Theorem 1.14.3 (Recursion for sets)**.** *Suppose that $<$ is a well-founded relation on a set u. Suppose that $G : u \times V \to V$ ("recursion rule"). Then there is a unique function $f : u \to V$ such that for all $x \in u$:*

$$f(x) = G(x, f\restriction\mathrm{pred}_<(x)).$$

Proof. Before we may prove this, we need another definition.

Definition. If $z \in u$, a function $\bar{f}$ is called a z-approximation if

1. $z \in \mathrm{dom}(\bar{f}) \subseteq u$.
2. For all $x \in \mathrm{dom}(\bar{f}), \mathrm{pred}_<(x) \subseteq \mathrm{dom}(\bar{f})$ and $\bar{f}(x) = G(x, \bar{f}\restriction\mathrm{pred}_<(x))$.

Claim 1. *Suppose that $z, z' \in u$, suppose that $\bar{f}$ is a z-approximation, $\bar{g}$ a z'-approximation. Let $v = \mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g})$. Then $\bar{f}\restriction v = \bar{g}\restriction v$.*

Proof. We show that $\bar{f}(x) = \bar{g}(x)$ for all $x \in v$, by induction along $(v, <)$.
Suppose that $y \in v$ and $\forall x < y\ \bar{f}(x) = \bar{g}(x)$. So $\bar{f}(x) = \bar{g}(x)$ for all $x \in v$. □

Claim 2. *If $z \in u$ and there is a z-approximation, then there is a $\subseteq$-least z-approximation.*

Proof. The intersection of all z-approximations works. □

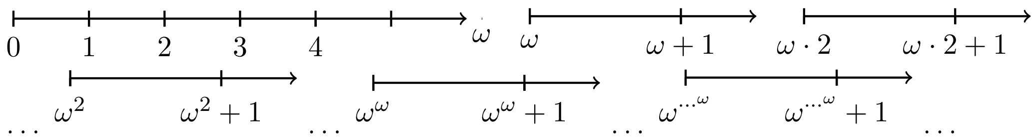
Claim 3. *For every $z \in u$, there is a z-approximation.*

Proof. We prove this by induction along $(u, <)$.
Suppose that for all $x < y$ there is an x-approximation. Let $\bar{f}_x$ denote the unique $\subseteq$-minimal x-approximation. Then $\bar{f} := \bigcup_{x<y} \bar{f}_x$ is a function by the first claim.
Moreover, $x \in \mathrm{dom}(\bar{f})$ for every $x < y$ and $\bar{f}$ is an x-approximation for all $x \in y$. Let $\bar{f}_y = \bar{f} \cup \{(y, G(y, \bar{f}))\}$. Then $\bar{f}_y$ is a y-approximation. □

Let $\bar{f}_z$ denote the unique $\subseteq$-minimal z-approximation for $z \in u$. Then $f = \bigcup_{z \in u} \bar{f}_z$ is a z-approximation for all $z \in u$, so f is as required. □

1.15 Ordinals

The notion of ordinals allows us to count beyond the natural numbers. We denote the ordinal that corresponds to the set of natural numbers by ω. Lecture 3 13th Oct

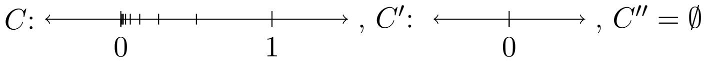


Note that these ordinals are all countable. The sequence $\{\omega, \omega \cdot 2, \ldots\}$ is a sequence of so called limit ordinals.

Example 1.15.1. If $C \subseteq \mathbb{R}$ is closed, let C' ("derivative of C") denote the set of all *non-isolated* points in C. We iterate the derivative along a well-order, by taking the intersection at limits:

$$C^{(0)} = C, C^{(1)} = C', \ldots, C^{(\omega)} = \bigcap_{n \in \mathbb{N}} C^{(n)}.$$

This process terminates at the maximal closed $D \subset C$ with no isolated points, the *perfect kernel* of C. Consider, for example, $C = \left\{x \in \mathbb{R} | x = 0 \vee \exists n \in \mathbb{N} \setminus \{0\}\, x = \frac{1}{n}\right\}$. Then $C' = \{0\}$ and $C'' = \emptyset$.



Definition 1.15.2.

1. A class A is called *transitive* if $\forall x \in A \forall y \in x\ y \in A$ (or, equivalently, $x \in y \wedge y \in A \Rightarrow x \in A$ for all sets x, y.)
2. Sets x, y are called $\in$-comparable if $x \in y$ or $y \in x$ or $x = y$.
3. A set x is called an *ordinal* if x is transitive and $(x, \in)$ is a linear order.
4. If α, β are ordinals, let $\alpha < \beta$ if $\alpha \in \beta$.
5. An ordinal of the form $\alpha + 1 = \alpha \cup \{\alpha\}$ is a *successor ordinal*.
6. An ordinal α is called a *limit ordinal* if $\alpha \neq 0$ and α is not a successor ordinal.
7. Let Ord denote the class of ordinals.
8. Let ω denote the $\subseteq$-least inductive set.

We denote ordinals by Greek miniscules $\alpha, \beta, \gamma, \ldots$

Lemma 1.15.3.

1. $0 \in \mathrm{Ord}$ *and* $\forall \alpha \in \mathrm{Ord}\ \alpha + 1 \in \mathrm{Ord}$.

2. Ord *is transitive.*

3. All $\alpha, \beta \in \mathrm{Ord}$ *are* $\in$*-comparable.*

4. Ord *is a proper class.*

5. If $x \subseteq \mathrm{Ord}$ *is a set, then* $\sup x := \bigcup x \in \mathrm{Ord}$.

Proof.

1. $0 = \emptyset \in \mathrm{Ord}$, because $\emptyset$ is trivially transitive and a linear order with $\in$. We have to show that for every $\alpha \in \mathrm{Ord}$, the successor $\alpha + 1$ is also transitive and $(\alpha + 1, \in)$ is a linear order. To show that $\alpha + 1$ is transitive, suppose that $\beta \in \alpha + 1$ and $\gamma \in \beta$. Note that $\alpha + 1 = \alpha \cup \{\alpha\}$. If $\beta \in \alpha$, then $\gamma \in \alpha$, by the transitivity of $\alpha \in \mathrm{Ord}$. If $\beta = \alpha$, then $\gamma \in \beta = \alpha \subseteq \alpha + 1$. Thus, $\alpha + 1$ is transitive. $(\alpha + 1, \in)$ is a linear order, because $\alpha + 1 = \alpha \cup \{\alpha\}$. $\square$
2. If $\alpha \in \beta \in \mathrm{Ord}$. We have $\alpha \subseteq \beta$, because β is transitive. So $(\alpha, \in)$ is linear. Suppose that $\delta \in \gamma \in \alpha \in \beta$. We have $\delta \in \alpha \vee \delta = \alpha \vee \alpha \in \delta$. By Foundation, $\delta \in \alpha$, since we would otherwise get an $\in$-circle. $\square$

3. Suppose that there is some ordinal α such that some β is not $\in$-comparable with α.
   Suppose that $\alpha_0 \in \alpha + 1$ is $\in$-minimal (by Foundation) such that some $\beta \in \mathrm{Ord}$ is not $\in$-comparable with α_0.
   Suppose that $\beta_0 \in \beta + 1$ is $\in$-minimal such that α_0, β_0 are not $\in$-comparable. We claim that $\alpha_0 = \beta_0$ by mutual inclusion:

$\alpha_0 \subseteq \beta_0$: Suppose that $\gamma \in \alpha_0$. Then γ is $\in$-comparable with β_0, by the minimality of α_0. If $\gamma = \beta_0$ then $\beta_0 \in \alpha_0$, contradiction. If $\beta_0 \in \gamma$, then $\beta_0 \in \alpha_0$, since α is transitive, contradiction. Thus, $\gamma \in \beta_0$.

$\beta_0 \subseteq \alpha_0$: Suppose that $\gamma \in \beta_0$. Then γ is $\in$-comparable with α_0, by minimality of β_0. As in the proof of the previous claim, $\gamma \in \alpha_0$.

   Hence, $\alpha_0 = \beta_0$, contradiction. □

4. Suppose that Ord is a set. As we have seen in 2. and 3., Ord is transitive and forms a linear order with $\in$. Thus, $\mathrm{Ord} \in \mathrm{Ord}$, contradicting Foundation. This is also known as the *Burali-Forti paradox*. □

5. Exercise. □

□

Lemma 1.15.4. *1. If $0 \in x \subseteq \omega$ and $\forall n \in x\, n + 1 \in x$, then $x = \omega$ (induction on natural numbers).*

2. ω is the least limit ordinal.

Proof. 1. Since x is an inductive set, and ω is the $\subseteq$-least inductive set, the claim holds. □

2. Since $\omega \cap \mathrm{Ord}$ is inductive, $\omega = \{0, 1, 2, \ldots\} \subseteq \mathrm{Ord}$. Then $(\omega, \in)$ is linear. To show that ω is transitive, let $x = \{n \in \omega | \forall m \in n\, m \in \omega\}$. Since $0 \in x$ and if $n \in x$, then $n + 1 \in x$, x is inductive, thus, $x = \omega$. Hence, ω is transitive.
   If $\omega = \alpha + 1$ for some $\alpha \in \mathrm{Ord}$, then $\alpha \in \omega$, and therefore $\alpha + 1 \in \omega$, contradicting Foundation.

   Claim. *If $\alpha \in \omega$ is a limit ordinal, then α is inductive.*

   Proof of the Claim. As an ordinal, α is a linear order. Suppose that $\beta \in \alpha$. If $\alpha \in \beta + 1$, then $\alpha \in \beta \in \alpha$, or $\alpha = \beta \in \alpha$, contradiction. If $\alpha = \beta + 1$, this would contradict the assumption that α is a limit ordinal. Therefore, $\beta + 1 \in \alpha$. □

   Since ω is the $\subseteq$-least inductive set, $\alpha = \omega$, contradiction.

□

Lemma 1.15.5. *1. Suppose that R is a well-founded relation, i.e. there is an R-minimal element and for all x, $\mathrm{pred}_R(x)$ is a set. Suppose x is a set (note that $\mathrm{pred}_\in(x) = x$).*
Then there is a $\subseteq$-least y with $x \subseteq y$ and $\forall z \in y\ \mathrm{pred}_R(z) \subseteq y$.

2. *If x is a set, there is a $\subseteq$-least set $\mathrm{tc}(x)$ ("**transitive closure** of x") with $x \subseteq y$ such that y is transitive.* *definition corrected 11th Nov*

Proof. 1. Let

$$f(0) = x, f(n+1) = f(n) \cup \bigcup_{z \in f(n)} \mathrm{pred}_R(z),$$

by recursion along $(\omega, <)$. The union in this equation is indeed a set: consider the function

$$f(n) \to V,$$
$$z \mapsto \mathrm{pred}_R(z).$$

By Replacement, $\{\mathrm{pred}_R(z) | z \in f(n)\}$ is a set, and applying Union to this set shows this claim.
Let

$$y = \bigcup_{n \in \omega} f(n).$$

□

2. Let $R = \in$ in 1. □
□

Lemma 1.15.6. *Suppose that R is a well-founded relation. Then any nonempty class A has an R-minimal element.*

Proof. Let $x \in A$. By Lemma 1.15.5 applied to $R \restriction A$ and $\{x\}$, there is some y such that $x \in y \subseteq A$ and $\forall z \in y \ \mathrm{pred}_{R \restriction A}(z) \subseteq y$.
By assumption on R, y has an R-minimal element z.
If z is not R-minimal in A, then there is some $z' \in A$ with $z' R z$. Then $z' \in y$. This contradicts the minimality of y. □

Theorem 1.15.7 (Induction for classes)**.** *Suppose that $<$ is a well-founded relation on a class A, $\varphi(x, y)$ a formula, v a set. If for all $y \in A$*

$$(\forall x < y \ \varphi(x, v)) \to \varphi(y, v)$$

then $\varphi(x, v)$ holds for all $x \in A$.

Proof. Analogous to the proof of Theorem 1.14.2 (induction for sets), using Lemma 1.15.3 (properties of ordinals). □

Theorem 1.15.8 (Recursion for classes)**.** *Suppose that $<$ is a well-founded relation on a class A and $G : A \times V \to V$ is a function ("recursion rule").*
Then there is a unique function $F : A \to V$ such that for all $x \in A$

$$F(x) = G(x, F \restriction \mathrm{pred}_<(x)).$$

Proof. Analogous to the proof of Theorem 1.14.3 (recursion for sets), using Theorem 1.15.7 (induction for classes). □

1.16 The Mostowski Collapse

Definition 1.16.1. A relation R on a class A is called *extensional* if for all $x, y \in A$:

$$\mathrm{pred}_R(x) = \mathrm{pred}_R(y) \to x = y.$$

For example, any linear order is extensional, while not all partial orders are extensional.

Theorem 1.16.2 (Mostowski's isomorphism theorem)**.** *Suppose that $<$ is a well-founded extensional relation on a class A.*
*Then there is a unique transitive class B ("**transitive collapse** of $(A,<)$" or "**Mostowski collapse** of $(A,<)$") and a unique isomorphism $\pi = \pi_{(A,<)} : (A,<) \to (B,\in)$ ("**collapsing map** of $(A,<)$"). The inverse map π^{-1} of the transitive collapse is also called the **monotone enumeration** of A.*

Proof. Let $\pi(y) \overset{G}{=} \{\pi(x) | x < y\}$ for $y \in A$, by recursion along $(A,<)$. Lecture 4 15th Oct
Let $B = \pi[A]$.

Claim. *π is injective.*

Proof. Suppose that z is $\in$-minimal such that there are $x \neq y$ with $\pi(x) = \pi(y) = z$.
Then $\mathrm{pred}_<(x) \neq \mathrm{pred}_<(y)$. Suppose that $u \in \mathrm{pred}_<(x) \setminus \mathrm{pred}_<(y)$.
Since $\pi(u) \in \pi(x) = \{\pi(v)|v < x\} = \pi(y) = \{\pi(v)|v < y\}$, $\pi(u) = \pi(v)$ for some $v < y$.
Then $u \neq< $ and $\pi(u) = \pi(v) \in z$. This contradicts the minimality of z. □

Claim. *π is an isomorphism.*

proof. If $x < y$, then $\pi(x) \in \pi(y)$. If $\pi(x) \in \pi(y) = \{\pi(z)|z < y\}$, then $\pi(x) = \pi(z)$ for some $z < y$. Then $x = z < y$. □

This proves the existence of π.

Claim. *$\pi[A] := B$ is transitive.*

Proof. Suppose that $x \in y \in B = \pi[A]$. Then $y = \pi(u)$ for some $u \in A$, so $y = \pi(u) = \{\pi(v)|v < u\}$. So $x \in \pi[A]$. □

Claim. *π is unique.*

Proof. Suppose that $\rho : (A,<) \to (C,\in)$ is another isomorphism, and C is transitive.
We prove this by induction that $\pi(x) = \rho(x)$ for all $x \in A$.
Suppose that $\pi(x) = \rho(x)$ for all $x < y \in A$.
Then $\pi(y) = \{\pi(x)|x < y\} = \{\rho(x)|x < y\} \subseteq \rho(y)$. Since ρ is an isomorphism, equality holds, $\{\rho(x)|x < y\} = \rho(y)$. □

□

Lemma 1.16.3. *The ordinals are exactly the transitive collapses of well-orders.*

Proof. By Theorem thm:mostowski (Mostowski's isomorphism theorem). □

Lemma 1.16.4. *If $(x,<),(y,<)$ are isomorphic well-orders, there is a unique isomorphism $f:(x,<)\to(y,<)$.*

Proof. $f=\pi_{(y,<)}^{-1}\circ\pi_{(x,<)}$. By composition of unique isomorphism, the diagram commutes.

$$\begin{tikzcd}
(x,<) \arrow[r, "\cong", "\exists! f"'] \arrow[d, "\exists!\pi_{(x,<)}"'] & (y,<) \arrow[d, "\exists!\pi_{(y,<)}"] \\
(\alpha,\in) \arrow[r, "\cong", "\mathrm{id}"'] & (\beta,\in)
\end{tikzcd}$$

□

1.17 Ordinal Arithmetic

Definition 1.17.1. For ordinals α,β, we define $\alpha+\beta,\alpha\cdot\beta$ by induction on β.

1. $\alpha+0:=\alpha,\alpha+(\beta+1):=(\alpha+\beta)+1=\alpha+\beta\cup\{\alpha+\beta\}$.
   For limit ordinals β, $\alpha+\beta:=\sup_{\gamma<\beta}(\alpha+\gamma)$.
2. $\alpha\cdot 0:=0,\alpha\cdot(\beta+1):=(\alpha\cdot\beta)+\alpha$.
   For limit ordinals β, $\alpha\cdot\beta:=\sup_{\gamma<\beta}(\alpha\cdot\gamma)$.

Definition 1.17.2. The lexicographical order on $\mathrm{Ord}^2:=\mathrm{Ord}\times\mathrm{Ord}$ is defined by $(\alpha,\beta)<_{\mathrm{lex}}(\gamma,\delta)$ if $\alpha<\gamma$ or ($\alpha=\gamma$ and $\beta<\delta$).

Exercise 1.17.3. $(\mathrm{Ord}^2,<_{\mathrm{lex}})$ *is a linear order.*

Lemma 1.17.4. *Suppose that $\alpha,\beta\in\mathrm{Ord}$.*

1. *There is a unique isomorphism $f_{\alpha,\beta}:(\alpha+\beta,\in)\to((\{0\}\times\alpha,\{1\}\times\beta),<_{\mathrm{lex}})$. It "glues" the two orders together.*
2. *There is a unique isomorphism $g_{\alpha,\beta}:(\alpha\cdot\beta,\in)\to(\beta\times\alpha,<_{\mathrm{lex}})$.*

Proof.
1. By induction on β. Clear for $\beta=0$.
   If $f_{\alpha,\beta}$ exists, let $f_{\alpha,\beta+1}=f_{\alpha,\beta}\cup\{(\alpha+\beta),(1,\beta)\}$ (max. el. of $(\{0\}\times\alpha)\cup(\{1\}\times\beta+1)$).
   For limits β, let $f_{\alpha,\beta}=\bigcup_{\gamma<\beta}f_{\alpha,\gamma}$.
2. By induction on β. Clear for $\beta=0$.
   If $g_{\alpha,\beta}$ exists, let $g_{\alpha,\beta+1}=g_{\alpha,\beta}\cup\{(\alpha\cdot\beta+\gamma,(\beta,\gamma))|\gamma<\alpha\}$.
   For limits β, let $g_{\alpha,\beta}=\bigcup_{\gamma<\beta}g_{\alpha,\gamma}$.

The uniqueness follows from Theorem 1.15.8 (recursion for classes). □

We sometimes write $+_{\mathrm{Ord}},\cdot_{\mathrm{Ord}}$ to explicitly refer to ordinal arithmetic (e.g. when talking about cardinal arithmetic, as is section 2.2).

1.18 The Von Neumann Hierarchy

Definition 1.18.1 (Rank)**.** By recursion along $(V, \in)$, we define

$$\operatorname{rank}(x) = \sup\{\operatorname{rank}(y) + 1 | y \in x\}.$$

Axiom (Power Set)**.** For any set x, there is a set $P(x)$, the *power set of* x, such that

$$\forall y (y \in P(x) \leftrightarrow y \subseteq x).$$

Definition 1.18.2. ZF is the axiom system ZF^- together with the Power Set Axiom.

Definition 1.18.3 (Von Neumann hierarchy)**.** By recursion along $(\mathrm{Ord}, \in)$, let

(i) $V_0 = \emptyset$.

(ii) $V_{\alpha+1} = P(V_\alpha)$.

(iii) $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for limits α.

Lemma 1.18.4. *Suppose that* $\alpha, \beta \in \mathrm{Ord}$.

1. V_α *is transitive.*
2. $V_\alpha \in V_\beta$ *if* $\alpha < \beta$.
3. $V_\alpha \cap \mathrm{Ord} = \alpha$.
4. $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$.

Proof.

1. The power set of a transitive set is transitive, and an increasing union of transitive sets is transitive (exercise).
2. By induction on $\beta > \alpha$. We have $V_\alpha \in V_{\alpha+1} = P(V_\alpha)$.
If $V_\alpha \in V_\beta \in V_{\beta+1}$, then $V_\alpha \in V_{\beta+1}$, since $V_{\beta+1}$ is transitive by 1.
3. By induction on α. We have $V_0 \cap \mathrm{Ord} = 0$.
Suppose that $V_\alpha \cap \mathrm{Ord} = \alpha$, so $\alpha \in V_{\alpha+1} \cap \mathrm{Ord}$, so $\alpha + 1 = \alpha \cup \{\alpha\} \subseteq V_{\alpha+1} \cap \mathrm{Ord}$.
If $\beta \in V_{\alpha+1} \cap \mathrm{Ord}$, then $\beta \subseteq V_\alpha \cap \mathrm{Ord} = \alpha$, so $\beta \in \alpha + 1$.
Therefore, $V_{\alpha+1} \cap \mathrm{Ord} = \alpha + 1$.
If α is a limit, then
$$V_\alpha \cap \mathrm{Ord} = \bigcup_{\beta<\alpha} (V_\beta \cap \mathrm{Ord}) = \bigcup_{\beta<\alpha} \beta = \sup_{\beta<\alpha} \beta = \alpha.$$
4. Suppose that x is $\in$-minimal with $x \notin \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$.
If $y \in x$, let $f(y)$ denote the least ordinal α such that $y \in V_\alpha$.
Let $\beta = \sup f[y]$. Then $x \subseteq V_\beta$ and $x \in V_{\beta+1}$, contradiction.

□

1.19 The Real Numbers

Definition 1.19.1. Let $\mathbb{N} = \omega, \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.
If $(m, n, k), (m', n', k') \in \mathbb{N}^2 \times \mathbb{N}^+$, let

$$(m, n, k) \sim (m', n', k') \text{ if } mk' + n'k = m'k + nk'$$

After multiplication is defined and the existence of multiplicative inverses of nonzero rationals is proved, this condition is equivalent to $\dfrac{m-n}{k} = \dfrac{m'-n'}{k'}$.

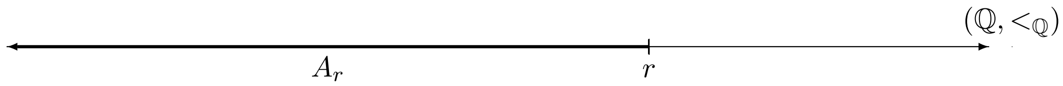
1. Let $\mathbb{Q} = (\mathbb{N}^2 \times \mathbb{N}^+ / \sim)$.
2. Let $[(m, n, k)]_\sim <_\mathbb{Q} [(m', n', k')]_\sim$ if $nk' + n'k < m'k + nk'$.
3. Let $[(m, n, k)]_\sim +_\mathbb{Q} [(m', n', k')]_\sim = [(mk' + m'k, nk' + n'k, kk')]_\sim$.
4. Let $[(m, n, k)]_\sim \cdot_\mathbb{Q} [(m', n', k')]_\sim = [(mm' + nn', mn' + m'n, kk')]_\sim$.
5. Let $0_\mathbb{Q} = [(0, 0, 1)]_\sim$ and $1_\mathbb{Q} = [(1, 0, 1)]_\sim$.

It can be checked that $(\mathbb{Q}, <_\mathbb{Q}, +_\mathbb{Q}, \cdot_\mathbb{Q}, 0_\mathbb{Q}, 1_\mathbb{Q})$ is an ordered field.

Definition 1.19.2. Let $\mathbb{R}$ denote the set of left halves A of Dedekind cuts in $(\mathbb{Q}, <_\mathbb{Q})$, i.e. a nonempty subset of $\mathbb{Q}$ with the following properties:

1. A is *downwards closed*: $\forall x \in A \forall y < x \; y \in A$.
2. A is *upwards bounded*: $\exists x \forall y \in A \; y \subseteq x$.
3. A has no maximal element.

For instance, $\mathbb{R} \ni r = \{q \in \mathbb{Q} | q < r\}$.



1. Let $x +_\mathbb{R} y = \{p +_\mathbb{Q} q \mid p \in x, q \in y\}$.
2. Let $0_\mathbb{R} = \{p \in \mathbb{Q} \mid p \leq_\mathbb{Q} 0_\mathbb{Q}\}$, $1_\mathbb{R} = \{p \in \mathbb{Q} \mid p \leq_\mathbb{Q} 1_\mathbb{Q}\}$.
3. Let $x <_\mathbb{R} y$ if $x \subsetneq y$.

To define the multiplication on $\mathbb{R}$, we first define $\sigma : \mathbb{R} \to P(\mathbb{Q})$ as follows. Addendum Lec5, 20th Oct

1. If $x \in \mathbb{R}$ and $x <_\mathbb{R} 0_\mathbb{R}$, let

$$\sigma(x) = \{p \in \mathbb{Q} \mid \exists q \in \mathbb{Q} \; (q <_\mathbb{Q} p \wedge q \notin x) \wedge p \leq 0\}.$$

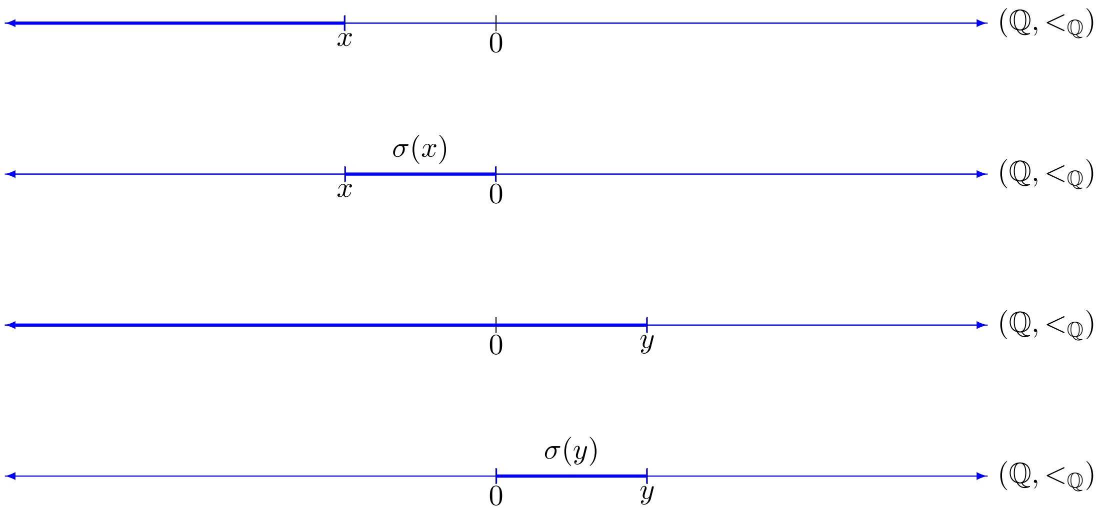
2. If $x \in \mathbb{R}$ and $x \geq_{\mathbb{R}} 0_{\mathbb{R}}$, let

$$\sigma(x) = \{p \in \mathbb{Q} \mid p \in y \wedge p \geq 0\}.$$

It can be checked that σ is injective, and that $\sigma(x) \cdot \sigma(y) := \{p \cdot_{\mathbb{Q}} q \mid p \in \sigma(x), q \in \sigma(y)\} \in ran(\sigma)$ for all $x, y \in \mathbb{R}$. Let

$$x \cdot_{\mathbb{R}} y := \sigma^{-1}[\sigma(x) \cdot_{\mathbb{R}} \sigma(y)].$$

It can be checked that $(\mathbb{R}, <_{\mathbb{R}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 0_{\mathbb{R}}, 1_{\mathbb{R}})$ is a complete ordered field.



2 Cardinals

Assuming the Axiom of Choice, we can associate to any set x the least ordinal with the same size as x. This ordinal is called the *cardinality* or *size* of x. In this section, we study these ordinals, the *cardinal numbers*, and operations on them: addition, multiplication, and exponentiation.

2.1 Cardinals and the Axiom of Choice

We can compare the size of two sets by the existence of injections between them.

Definition 2.1.1. Suppose that x, y are sets. We write $x \preceq y$ if there is an injection $f : x \to y$. Lecture 5 20th Oct

For example $\omega + 1 \preceq \omega$, i.e. there is an injective function $f : \omega + 1 \to \omega$. The relation $\preceq$ is transitive. The existence of injections between two sets in both directions implies the existence of a bijection by the next result.

Lemma 2.1.2 (Bernstein-Cantor)**.** *Suppose that a, b are sets and $f : a \to b$ and $g : b \to a$ are injective.*
Then there is a bijection $h : a \to b$.

Proof. Let $c = f[a] \subseteq b$, let $h = f \circ g : b \to c$.
Then h is injective.
We define by induction on $n \in \omega$:

$$b_0 = b, \qquad c_0 = c$$
$$b_{n+1} = h[b_n], \qquad c_{n+1} = h[c_n]$$

Then $b_0 = b \supseteq c = c_0 \supseteq h[b] = b_1$. Then $b_n \supseteq c_n \supseteq h[b_n] = b_{n+1}$ for all n, by induction on n.
Then b is partitioned into (i.e. is a disjoint union of) the sets

$$u = \bigcap_{n\in\omega} b_n = \bigcap_{n\in\omega} c_n,$$
$$v = \bigcup_{n\in\omega} (b_n \setminus c_n),$$
$$w = \bigcup_{n\in\omega} (c_n \setminus b_{n+1}).$$

Also, c is partitioned into the sets u, v', and w, where

$$v' = \bigcup_{n\in\omega} (b_{n+1} \setminus c_{n+1}).$$

Since h is injective, $h[b_n \setminus c_n] = h[b_n] \setminus h[c_n] = b_{n+1} \setminus c_{n+1}$.
So $h{\restriction}v : v \to v'$ is bijective.
Let $i = \mathrm{id}_{u\cup w} \cup (h{\restriction}v)$. Then $i : b \to c$ is bijective. $\square$

Lemma 2.1.3. *Suppose that a is a set.*

1. *There is an injection $f : a \to P(a)$.*
2. *There is a bijection $g : P(a) \to {}^a2$.*
3. ***Cantor's Theorem:** There is no injection $h : P(a) \to a$.*

Proof. 1. Let $f : a \to P(a), f(x) = \{x\}$. This is injective by Extensionality.

2. Let $g : P(a) \to {}^a2$, for $y \in a, g(x)(y) = 1$ if $y \in x$ and $g(x)(y) = 0$ otherwise.

3. Suppose that $h : {}^a2 \to a$ is an injection. We can assume that h is a bijection by 1. and the Bernstein-Cantor Theorem 2.1.2. There is a unique $x \in {}^a2$ such that $x(i) \neq h^{-1}(i)(i)$ for all $i \in a$. Then $x \neq h^{-1}(i)$ for all $i \in a$, contradiction.

□

Lemma 2.1.4. *Suppose that $\alpha \in$ Ord.*

1. *If there is a surjection $f : \alpha \to x$, then there is an injection $g : x \to \alpha$.*
2. *If $g : x \to \alpha$ is an injection, then there is a bijection $h : x \to \beta$, for some $\beta \leq \alpha$.*

Proof. 1. Let $g : x \to \alpha$, where $g(y), y \in x$, is the least $\beta < \alpha$ with $f(\beta) = y$.
In fact, $f \circ g = \mathrm{id}_x$.

2. Let $h = \pi \circ g$, where $\pi = \pi_{(g[x],<)}$ is the transitive collapse of $(g[x], <)$.

□

Definition 2.1.5. 1. An ordinal α is called a *cardinal (number)* if for all $\beta < \alpha$, there is no injection $f : \alpha \to \beta$.

2. Let Card denote the class of cardinals.

3. If x is a set, let $|x| = \bar{\bar{x}} = \mathrm{card}(x)$ denote the least ordinal α such that there is a bijection $f : \alpha \to x$, if this exists.

We write Greek miniscules $\kappa, \lambda, \mu, \ldots$ for cardinals.

Lemma 2.1.6. *An ordinal α is a cardinal if and only if there is some set x with $|x| = \alpha$.*

Proof. If $\alpha \in$ Card, then $|\alpha| = \alpha$.
Suppose that $|x| = \alpha$ and there is an injection $f : \alpha \to \beta$ for some $\beta < \alpha$.
Then by the Bernstein-Cantor Theorem 2.1.2, there is a bijection $g : \alpha \to \beta$.
Then $|x| \leq \beta < \alpha$, contradicting the assumption $|x| = \alpha$, i.e. the minimality of α. □

Definition 2.1.7. 1. A set x is called *finite* if there is an injection $f : x \to n$ for some $n \in \omega$.

2. A set x is called *infinite* if it it not finite.

3. A set x is called *countable* if there is an injection $f : x \to \omega$.

4. A set x is called *uncountable* if it is not countable.

Lemma 2.1.8. *The following are equivalent for an ordinal α.*

1. $\alpha < \omega$.
2. *Every injective $f : \alpha \to \alpha$ is surjective.*
3. *Every surjective $f : \alpha \to \alpha$ is injective.*

Proof.

1. $\Rightarrow$ 2.: By induction on $\alpha = n < \omega$. This holds for $n = 0$.
Suppose this holds for some n.
Suppose that $f : n + 1 \to n + 1 = n \cup \{n\}$ is injective, but not surjective.
We assume that $f(n) = n$ by switching two values of f.
Then $f{\restriction}n : n \to n$ and this is injective, but not surjective, contradicting the induction assumption.

2. $\Rightarrow$ 1.: Suppose that $\alpha \geq \omega$.
Let $f : \alpha \to \alpha, f(n) = n + 1$ for $n \in \omega$ and $f(\beta) = \beta$ for $\beta \geq \omega$.
Then f is injective, but not surjective.

1. $\Rightarrow$ 3.: By induction on $n = \alpha < \omega$.
This holds for $n = 0$. Suppose that the claim holds for n.
Suppose that $f : n + 1 \to n + 1$ is surjective, but not injective.
We can assume that $f(n) = n$ and that $\mathrm{range}(f{\restriction}n) = n$.
Then $f{\restriction}n : n \to n$ is surjective, but not injective, contradicting the induction assumption.

3. $\Rightarrow$ 1.: Suppose that $\alpha \geq \omega$. Let $g : \alpha \to \alpha, g(0) = 0, g(n + 1) = n$ for $n < \omega$, $g(\beta) = \beta$ for $\beta \geq \omega$. Then g is surjective, but not injective.

□

Definition 2.1.9. If $(x, <_0)$ is a well-order and $(\alpha, \in)$ is its transitive collapse, then α is also called the *order type* $\mathrm{type}(x, <_0) = \mathrm{otp}(x, <_0)$ of $(x, <_0)$.

Lemma 2.1.10. *For any cardinal κ there is some cardinal $\lambda > \kappa$.*

Proof. If $\kappa = n < \omega$, then $\lambda := n + 1$ is a cardinal by the previous lemma. Suppose that $\kappa \geq \omega$ and let

$$\lambda = \sup\{\mathrm{type}(\kappa, <_0) \mid (\kappa, <_0) \text{ is a well-order }\}.$$

denote the supremum of all enumerations of κ in different order types. Then $\lambda \in \mathrm{Ord}$ by the Power Set Axiom and the Replacement Scheme. Since $(\kappa, \in)$ is a wellorder, we have $\kappa \leq \lambda$. Moreover $|\alpha| \leq \kappa$ for any $\alpha < \lambda$ by the definition of λ.

We claim that λ is a cardinal. Otherwise there is an injection $f : \lambda \to \alpha$ for some $\alpha < \lambda$. Since $|\alpha| \leq \kappa$, there is also an injection $g : \lambda \to \kappa$. This implies that there is a bijection $h : \kappa \to \lambda$ by the Bernstein-Cantor Theorem. For $\alpha, \beta < \kappa$, let $\alpha <_0 \beta$ if $h(\alpha) \in h(\beta)$. Then $(\kappa, <_0)$ is isomorphic to $(\lambda, \in)$, i.e. $\mathrm{type}(\kappa, <_0) = \lambda$.

Claim. *There is a well-order* $(\kappa, <_1)$ *with* $\text{type}(\kappa, <_1) = \lambda + 1$.

Proof. Let $f\colon \kappa \to \kappa$, $f(n) = n+1$ for $n \in \omega$, $f(\alpha) = \alpha$ for $\alpha \geq \omega$. Let $x <_1 y$ if $y = 0$ or $f(x) <_0 f(y)$. Then $\text{type}(\kappa, <_1) = \lambda + 1$. □

Then $\lambda + 1 \leq \lambda$, contradiction. □

Lecture 6
22nd Oct

Lemma 2.1.11. *1. Suppose that* $x \subseteq \text{Card}$, *then* $\sup(x) \in \text{Card}$.

2. Every infinite cardinal is a limit ordinal.

Proof. Exercise. Hint for 2.: Construct a bijection $\alpha + 1 \leftrightarrow \alpha$ for α an ordinal. □

Lemma 2.1.12. *1.* $\forall n \in \omega : n \in \text{Card}$.

2. $\omega \in \text{Card}$.

Proof. 1. By Lemma 2.1.8.

2. By 1. and Lemma 2.1.11 1. $(\omega \subseteq \text{Card}, \sup \omega = \bigcup \omega = \bigcup_{n \in \omega} n = \omega \in Card)$.

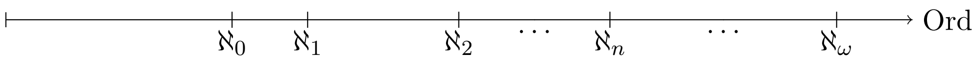
□

Definition 2.1.13. 1. If $\alpha \in \text{Ord}$, let α^+ denote the least cardinal λ with $\alpha < \lambda$.

2. We define $\aleph\colon \text{Ord} \to \text{Ord}$ ("*alef*"-function) by recursion on $\alpha \in \text{Ord}$.
   (i) $\aleph(0) := \aleph_0 := \omega_0 := \omega$.
   (ii) $\aleph(\alpha+1) := \aleph_{\alpha+1} := \omega_{\alpha+1} := \aleph_\alpha^+$.
   (iii) For limit ordinals α, $\aleph(\alpha) := \aleph_\alpha := \omega_\alpha := \sup_{\beta<\alpha} \aleph_\beta$.

3. A cardinal is called a *successor cardinal* if it is ofthe form κ^+ for some $\kappa \in \text{Card}$.

4. A cardinal is called a *limit cardinal* if it is nonzero and not a successor cardinal.



The size of the gap between the $\aleph$ numbers is strictly increasing, e.g. it is ω_1 between $\aleph_0$ and $\aleph_1$ and ω_2 between $\aleph_1$ and $\aleph_2$. I.e. $\text{type}(\omega_1 \setminus \omega) = \omega_1$.

For example, ω_1 is the length of the Borel hierarchy, the supremum of the ranks of countable trees, and the supremum of the Cantor-Bendixson ranks of closed sets of reals.

Lemma 2.1.14 (Cantor's paradox)**.** Card *is a proper class.*

Proof. Exercise. □

Lemma 2.1.15. *Every infinite cardinal is of the form* $\aleph_\alpha$ *for some* $\alpha \in \text{Ord}$.

Proof. Exercise. □

Axiom (Choice)**.** For every set x and every function $f : x \to V$ with $f(y) \neq \emptyset$ for all $y \in x$, there is a *choice function* $g : x \to V$ with $g(y) \in f(y)$ for all $y \in x$.

Definition 2.1.16. ZFC denotes the axiom system ZF together with the Axiom of Choice. These are often called the *axioms of set theory.*

Lemma 2.1.17. *$|a| \leq |b|$ if and only if there is a surjection $f : b \to a$.*

Proof. Exercise. This also uses the Axiom of Choice. □

Theorem 2.1.18.

1. ***Well-ordering Theorem:*** *Every set can be well-ordered.*
2. *For every set x, $|x|$ is defined.*

Proof. 2. follows from 1.
To prove 1., consider a choice function for $\mathrm{id}_{P(x)\setminus\{\emptyset\}}$, i.e.

$$g : P(x) \setminus \{\emptyset\} \to x, \text{ and } g(y) \in y \text{ for all } y \in P(x) \setminus \{\emptyset\}.$$

We define by recursion along $(\mathrm{Ord}, <)$ a function $h : \mathrm{Ord} \to V$ by

(i) $h(\alpha) = g(x \setminus h[\alpha])$ if $x \setminus h[\alpha] \neq \emptyset$.

(ii) $h(\alpha) = x$ otherwise.

Claim. *$h(\alpha) = x$ for some $\alpha \in \mathrm{Ord}$.*

Proof. Suppose not. We claim that h is injective.
If $\alpha < \beta \in \mathrm{Ord}$, then $h(\beta) \in x \setminus h[\beta]$, so $h(\alpha) \neq h(\beta)$.
Then $h[\mathrm{Ord}]$ is a set by the Separation Axiom.
Then $h^{-1} : h[\mathrm{Ord}] \to \mathrm{Ord}$ is surjective, so Ord is a set by Replacement, contradiction. □

Let α be least with $h(\alpha) = x$.

Claim. *$h{\restriction}\alpha$ is injective.*

Proof. If $b < \gamma < \alpha$, then $h(\gamma) \in x \setminus h[\gamma]$, but $h(\beta) \in h[\gamma]$, so $h(\beta) \neq h(\gamma)$. □

Claim. *$h(\beta) \in x$ for all $\beta < \alpha$.*

Proof. By minimality of α. □

Claim. *$h{\restriction}\alpha : \alpha \to x$ is surjective.*

Proof. Since $h(\alpha) = x$, so $x \setminus h[\alpha] = \emptyset$. □

This proves the theorem. □

Lemma 2.1.19. *Suppose $x, y \in V$. Then $x \preceq y$ if and only if $|x| \leq |y|$.*

Proof. Let $\kappa = |x|, \lambda = |y|$.
If $x \preceq y$, then $\kappa \preceq \lambda$. Since $\kappa \in \mathrm{Card}, \kappa \leq \lambda$.
If $\kappa \leq \lambda$, then $\kappa \preceq \lambda$ and $x \preceq y$. □

2.2 Cardinal Arithmetic

We first prove some more things about ordinal arithmetic.

Definition 2.2.1.
1. A function $f : \mathrm{Ord} \to \mathrm{Ord}$ is called *strictly monotone* or *strictly increasing* if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$, for all $\alpha, \beta \in \mathrm{Ord}$.
2. A function $f : \mathrm{Ord} \to \mathrm{Ord}$ is called *weakly monotone* or *weakly increasing* if $\alpha < \beta$ implies that $f(\alpha) \leq f(\beta)$ for all $\alpha, \beta \in \mathrm{Ord}$.
3. If $(\gamma_\alpha)_{\alpha<\beta}$ is a sequence in Ord where β is a limit ordinal, let $\lim_{\alpha<\beta} \gamma_\alpha = \gamma$ if for every $\delta < \gamma$, there is some $\eta < \beta$ such that for all ζ with $\eta < \zeta < \beta$, $\delta < f(\gamma_\zeta) \leq \gamma$.
4. A function $f : \mathrm{Ord} \to \mathrm{Ord}$ is called *continuous* if $\lim_{\beta<\alpha} f(\gamma_\beta) = f(\lim_{\beta<\alpha} \gamma_\beta)$ for all limit ordinals α and all sequences $(\gamma_\beta)_{\beta<\alpha}$ such that $\lim_{\beta<\alpha} \gamma_\beta$ exists.

corrected definitions

For weakly monotone functions f, the condition that f is continuous is equivalent to the condition that $\sup_{\beta<\alpha} f(\gamma_\beta) = f(\sup_{\beta<\alpha} \gamma_\beta)$ for all limit ordinals α and all sequences $(\gamma_\beta)_{\beta<\alpha}$.

Example 2.2.2.
- The $\aleph$-function (it is continuous since $\sup_{\beta<\alpha} \aleph_\beta = \aleph_\alpha$ for limits α).
- $a_\alpha : \mathrm{Ord} \to \mathrm{Ord}, a_\alpha(\beta) = \alpha + \beta$.
- $m_\alpha : \mathrm{Ord} \to \mathrm{Ord}, m_\alpha(\beta) = \alpha \cdot \beta$.

Lemma 2.2.3. *For $\alpha, \beta \in \mathrm{Ord}$, there are unique ordinals γ, δ with $\alpha = \beta \cdot \gamma + \delta$ and $\delta < \beta$.*

Proof. Let $\gamma = \sup\{\eta \mid \beta \cdot \eta \leq \alpha\}$. Then $\beta \cdot \gamma \leq \alpha$.

Claim. *Every $\eta \geq \beta$ is of the form $\beta + \zeta$ for some $\zeta \in \mathrm{Ord}$.*

Proof. By induction on $\eta \geq \beta$. □

Let $\alpha = \beta \cdot \gamma + \delta$, Then $\delta < \beta$ by definition of γ.

Claim. *γ, δ are unique.*

Proof. Suppose that $\alpha = \beta \cdot \gamma + \delta = \beta \cdot \eta + \zeta$ and $\delta, \zeta < \beta$.
Assume $\gamma < \eta$. Then $\alpha = \beta \cdot \gamma + \delta < \beta \cdot (\gamma + 1) \leq \beta \cdot \eta + \zeta = \alpha$, contradiction.
Thus, $\gamma = \eta$.
Then $\beta \cdot \gamma = \beta \cdot \eta$. Since $+$ is strictly monotone in the second argument, $\delta = \zeta$. □

□

From now on, we write $+_{\mathrm{Ord}}$ and $\cdot_{\mathrm{Ord}}$ for ordinal addition and multiplication, resp.

Definition 2.2.4. Suppose that $\kappa, \lambda \in \mathrm{Card}$.

1. $\kappa + \lambda := |x \cup y|$, where x, y are disjoint with $|x| = \kappa$, $|y| = \lambda$.
2. $\kappa \cdot \lambda := |\kappa \times \lambda|$.
3. $\kappa^\lambda := |{}^\lambda\kappa| = |\{f \mid f : \lambda \to \kappa\}|$.

Lemma 2.2.5. *For all $m, n \in \omega$, $m + n = m +_{\text{Ord}} n$ and $m \cdot n = m \cdot_{\text{Ord}} n$.*

Proof. Exercise. □

Lemma 2.2.6. *If $\kappa \geq \omega, \lambda$ are cardinals, then $\kappa + \lambda = \max\{\kappa, \lambda\}$.*

Proof. Suppose that $\lambda = \max\{\kappa, \lambda\}$.
Let

$$\begin{aligned} f : \big((\{0\} \times \kappa) \cup (\{1\} \times \lambda)\big) &\to \lambda \\ f(0, \alpha) = 2 \cdot_{\text{Ord}} \alpha, \qquad f(1, \alpha) &= 2 \cdot_{\text{Ord}} \alpha + 1. \\ \text{These are distinct by Lemma 2.2.3} \end{aligned}$$

If $\alpha < \lambda$, then $2 \cdot_{\text{Ord}} \alpha < \lambda$ by induction on α, since λ is a limit ordinal.
Hence, f is an injection. Therefore, $\kappa + \lambda = \lambda$, by Bernstein-Cantor. □

We now introduce the Gödel ordering on pairs of ordinals to characterise cardinal multiplication. We don't use the lexicographical order, because it is not well-founded, i.e. the class of predecessors of pairs of ordinals can be a proper class.

Definition 2.2.7 (Gödel ordering)**.** Suppose $(\alpha, \beta), (\alpha', \beta') \in \text{Ord}^2$.
Let $(\alpha, \beta) \lhd (\alpha', \beta')$ if

(i) $\max\{\alpha, \beta\} < \max\{\alpha', \beta'\}$ or

(ii) $\max\{\alpha, \beta\} = \max\{\alpha', \beta'\}$ and $(\alpha, \beta) <_{\text{lex}} (\alpha', \beta')$.

Lemma 2.2.8. $(\text{Ord}^2, \lhd)$ *is a well-order.*

Proof. For any $(\alpha, \beta) \in \text{Ord}^2, \text{pred}_\lhd(\alpha, \beta) = \{(\gamma, \delta) \mid (\gamma, \delta) \lhd (\alpha, \beta)\} \subseteq (\max\{\alpha, \beta\} + 1)^2$ is a set by Problem 10.
Suppose that $x \subseteq \text{Ord}^2$.
Let γ be $\in$-minimal such that $\gamma = \max\{\alpha, \beta\}$ for some $(\alpha, \beta) \in x$.
Let α be $\in$-minimal such that $\gamma = \max\{\alpha, \beta\}$ and $(\alpha, \beta) \in x$ for some β.
Let β be $\in$-minimal such that $\gamma = \max\{\alpha, \beta\}$ and $(\alpha, \beta) \in x$.
Then, (α, β) is $\lhd$-minimal in x. □

Definition 2.2.9 (Gödel pairing)**.** Let G denote the collapsing map of $(\text{Ord}, \lhd)$. Lecture 7 27th Oct

Lemma 2.2.10. *1. $G : \text{Ord}^2 \to \text{Ord}$ bijective.*

*2. If $\kappa \geq \omega$ is a cardinal, $G{\restriction}(\kappa^2) : \kappa^2 \to \kappa$ is bijective.
Hence, $\kappa \cdot \kappa = \kappa$.*

Proof. 1. Let A denote the transitive collapse of $(\mathrm{Ord}^2, \lhd)$.

Claim. $A \subseteq \mathrm{Ord}$.

Proof. For any $x \in A$, $(x, \in) = (\mathrm{pred}_{\in}(x), \in) \cong (G^{-1}(x), \lhd)$, $G^{-1}(x) \in \mathrm{Ord}^2$, is a well-order.
To see that x is transitive, suppose that $z \in y \in x \in A$.
Since A is transitive, $z \in A$. Since $(A, \in)$ is a linear order, z and x are $\in$-comparable, so $x \in z$ or $x = z$ or $z \in x$.
We have $z \neq x$, since otherwise $z \in y \in z$, contradicting the Foundation Axiom.
The same argument applies to $x \in z$. Then $z \in x$ and x is therefore an ordinal. □

A is a proper class, because Ord^2 is a proper class, and $G : \mathrm{Ord}^2 \to A$ is bijective.
So $\bigcup A = \mathrm{Ord}$. Since A is transitive, $A = \mathrm{Ord} = \bigcup A$.

2. We prove the claim by induction along $(\mathrm{Card} \setminus \omega, \in)$.
Suppose that $\kappa = \omega$. We have $|G[n \times n]| = n \cdot n = n \cdot_{\mathrm{Ord}} n$, by Problem 13.
We have $G[n \times n] = G(0, n) \in \mathrm{Ord}$.
So $G[n \times n]$ is a finite ordinal, i.e. $G[n \times n] \in \omega$.
Then $G[\omega \times \omega] = \bigcup_{n \in \omega} G[n \times n]$,
so $G[\omega \times \omega] = \omega$, because $\sup \bigcup_{n \in \omega} G[n \times n] = \omega$.
Suppose that $\kappa > \omega$ is a cardinal and that $G[\lambda \times \lambda] = \lambda$ for all $\lambda \in \mathrm{Card} \setminus \omega$ with $\lambda < \kappa$.
By Problem 12, κ is a limit ordinal.
By the induction hypothesis, for α an ordinal with $\omega \leq \alpha < \kappa$, we have

$$|G[\alpha \times \alpha]| = |\alpha| \cdot |\alpha| \overset{\text{ind.hyp.}}{=} |\alpha| < \kappa.$$

We have $G[\alpha \times \alpha] = G(0, \alpha) \in \mathrm{Ord}$.
Also, we have $G(0, \alpha) \geq \alpha$, by induction.
Then $G[\kappa \times \kappa] = \bigcup_{\alpha < \kappa} G[\alpha \times \alpha] = \kappa$.
Therefore, $\kappa \cdot \kappa = \kappa$. (For ordinals, this is not true in general.)

□

Lemma 2.2.11. *1. $|\mathbb{R}| = |{}^{\omega}2| = 2^{\omega}$.*

2. The set $\mathcal{C}(\mathbb{R}, \mathbb{R})$ of continuous functions of $\mathbb{R}$ has size 2^{ω}.

Proof. 1. We have $\mathbb{R} \subseteq P(\mathbb{Q})$, so $|\mathbb{R}| \leq |P(\mathbb{Q})| = |P(\omega)| = |{}^{\omega}2| = 2^{\omega}$.
We have the following fact from analysis.
If $I_n = [a_n, b_n] \subseteq \mathbb{R}, a_n < a_{n+1} < b_{n+1} < b_n$ and $\lim_{n \to \infty} b_n - a_n = 0$, then $\bigcap_{n \in \omega} I_n$ has a unique element.
Let ${}^{<\alpha}X = \bigcup_{\beta < \alpha}^{\beta} X$.
We construct $(I_s)_{s \in {}^{<\omega}2}$ recursively along $({}^{<\omega}2, \subsetneq)$, such that

(i) $I_s = [a_s, b_s] \subseteq \mathbb{R}$.

(ii) $b_s - a_s < \frac{1}{2^n}$, for $s \in^n 2$.

(iii) $a_s < a_{s^n(0)} < b_{s^n(0)} < a_{s^n(1)} < b_{s^n(1)} < b_s$.

If $x \in^\omega 2$, let $f(x)$ denote the unique elemen of $\bigcap_{n\in\omega} I_{x\restriction n}$.

2. For every $x \in \mathbb{R}$, let $c_x : \mathbb{R} \to \mathbb{R}$ denote the constant function with value x. Then $c_x \neq c_y$ for $x \neq y$.
We identify $p \in \mathbb{Q}$ with $\{q \in \mathbb{Q} \mid q \leq_{\mathbb{Q}} p\} \in \mathbb{R}$.
Then $\mathbb{Q}$ is dense in the reals, $\forall x \in \mathbb{R} \forall n \exists y \in \mathbb{Q}\, |x - y| < \frac{1}{2^n}$.
Hence every continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniquely determined by $f\restriction\mathbb{Q}$.
There are $\leq \left|{}^{\mathbb{Q}}\mathbb{R}\right| = \left|{}^{\omega}({}^{\omega}2)\right| = |{}^{\omega\times\omega}2| = |{}^{\omega}2| = 2^\omega$.

□

Definition 2.2.12. 1. A set $A \subseteq P(x)$ is called a *σ-algebra* if $\emptyset \in A, x \in A$ and A is closed under complements and countable unions.

2. A set $x \subseteq \mathbb{R}$ is called a *Borel set* if it is an element of the $\subseteq$-least σ-algebra on $\mathbb{R}$ such that the open interval $(a, b), a, b \in \mathbb{Q}$ is in this σ-algebra. This σ-algebra is also called the *Borel σ-algebra* on $\mathbb{R}$.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is called *Borel measurable* if $f^{-1}[(a, b)]$ is a Borel set for all $a, b \in \mathbb{Q}$.

Definition 2.2.13 (Borel codes)**.** 1. A *well-founded labeled tree* is a pair (t, f) such that

(i) $t \subseteq {}^{<\omega}\omega, t \neq \emptyset$.

(ii) t is closed under initial segments, i.e. $\forall v \in t \forall u \subseteq v\, (u \in {}^{<\omega}\omega \to u \in t)$.

(iii) t has no infinite branches, i.e. $x \in {}^{\omega}\omega$ such that $x\restriction n \in t$ for all n.
(t with reverse ordering is well-founded.)

(iv) $f : \text{end}(t) \to \{(a, b) \mid a, b \in \mathbb{Q}\}$, where $\text{end}(t)$ denotes the set of $u \in t$ such that there is no $v \in t$ with $u \subsetneqq v$.

typographical errors in 2. corrected 11th Nov

2. Suppose that (t, f) is a well-founded labeled tree on ω.
Note that $(t, \supsetneqq)$ is well-founded by a Problem on the next sheet.
We define by recursion along $(t, \supsetneqq)$ for $s \in t$.

$$B_{t,f}(s) = \begin{cases} f(s), & \text{if } s \in \text{end}(t). \\ \mathbb{R} \setminus B_{t,f}(s^\frown(i)), & \text{if } s \in 2^n \text{ for some odd } n \\ & \text{ and } i \text{ is least with } s^\frown(i) \in t. \\ \bigcup_{i\in\omega} B_{t,f}(s^\frown(n_i)), & \text{if } s \in 2^n \text{ for some even } n \\ & \text{ and } \{s^\frown(n_i) \mid i \in \omega\} \text{ are the successors of } s \text{ in } t. \end{cases}$$

where $(s_0, ..., s_n)^\frown(s) := (s_0, ..., s_n, s)$. Let $B_{t,f} = B_{t,f}(\emptyset)$.

Lemma 2.2.14. *Let T denote the set of well-founded labeled trees on ω.*
Let

$$\mathcal{B} := \{B_{t,f} \mid (t,f) \in T\}.$$

1. *$\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$.*
2. *$|\mathcal{B}| = 2^\omega$.*
3. *Th set of Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ has size 2^ω.*
4. *The set of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ has size $2^{(2^\omega)}$.*

Proof.
1. Every set $B_{t,f}(s)$ for $s \in t$ is a Borel set by induction on $(t, \supsetneq)$.
Also, $\mathcal{B}$ is a σ-algebra which contains all open intervals $(a,b) \subseteq \mathbb{R}$ with $a,b \in \mathbb{Q}$.
Since the Borel σ-algebra is the $\subseteq$-least such σ-algebra, $\mathcal{B}$ is the σ-algebra of Borel sets.
2. We have $|\mathcal{B}| \geq 2^\omega$, because $|\mathbb{R}| = 2^\omega$ and $\{x\} \in \mathcal{B}$ for any $x \in \mathbb{R}$.
We have $|P\left({}^{<\omega}\omega\right)| \leq |{}^\omega 2| = 2^\omega$.
and $\left|{}^{({}^{<\omega}\omega)}\omega\right| = |{}^\omega\omega| \leq \left|{}^\omega({}^\omega 2)\right| = |{}^{\omega\times\omega}2| = |{}^\omega 2| = 2^\omega$.
Moreover, $|\mathcal{B}| \leq 2^\omega$, since there are at most $\left|\underbrace{P({}^{<\omega}\omega)}_{\ni t} \times \underbrace{{}^{({}^{<\omega}\omega)}\omega}_{\ni f}\right| \leq 2^\omega \cdot 2^\omega = 2^\omega$ many well-founded labeled trees on ω.
3. Every Borel measurable $f : \mathbb{R} \to \mathbb{R}$ is determined by the sets $f^{-1}[(a,b)]$, where $a,b \in \mathbb{Q}$. There are at most $|{}^\omega|\mathcal{B}|| = \left|{}^\omega({}^\omega 2)\right| = |{}^{\omega\times\omega}2| = |{}^\omega 2| = 2^\omega$.
4. We use the following fact from analysis.
There is a set $A \subseteq \mathbb{R}$ with $|A| = |\mathbb{R}|$ and measure 0 (e.g. the Cantor discontinuum).
Then every $B \subseteq A$ has measure 0, and there are $|P(A)| = 2^{(2^\omega)}$ many such sets B.
The characteristic functions of these sets are Lebesgue measurable.

□

2.3 Infinite Sums and Products

Definition 2.3.1. 1. A *sequence* is a function $f : \alpha \to V$ for some $\alpha \in \mathrm{Ord}$. Lecture 8 29th Oct

2. We also write a function $f : s \to V$ as $\big(f(i)\big)_{i \in s}$.

Definition 2.3.2. Suppose that s is a set and $\kappa_i \in \mathrm{Card}$ for $i \in s$.

1. Let
$$\sum_{i\in s} \kappa_i = \left|\bigcup_{i\in s} X_i\right|,$$
where $|X_i| = \kappa_i$ for $i \in s$ and $X_i \cap X_j = \emptyset$ for $i \neq j$.

2. Let
$$\prod_{i\in s}\kappa_i = \left|\prod_{i\in s} X_i\right|,$$
where $|X_i| = \kappa_i$ for $i \in s$ and $\prod_{i\in s} X_i = \{f : s \to \bigcup_{i\in s} X_i \mid \forall i \in s\; f(i) \in X_i\}$.

Lemma 2.3.3. *Suppose that s is an infinite set and $\kappa_i \in \mathrm{Card}$ with $\kappa_i \geq 1$ for $i \in s$, $\kappa = \sup_{i\in s}\kappa_i$.*

1. *$\sum_{i\in s}\kappa_i = |s|\cdot\kappa = \max\{|s|,\kappa\}$.*
2. *If $|s| \leq \kappa$, then $\sum_{i\in s}\kappa_i = \kappa$.*
3. *If $\kappa_i \geq 2$ for all $i \in s$, then $\sum_{i\in s}\kappa_i \leq \prod_{i\in s}\kappa_i$.*

Proof.
1. We have $\sum_{i\in s}\kappa_i \leq |s|\cdot\kappa$.
Moreover, $|s| \leq \sum_{i\in s}\kappa_i$, because $\kappa_i \geq 1$ for each $i \in s$.
Also, $\kappa = \sup_{i\in s}\kappa_i \leq \sum_{i\in s}\kappa_i$.
2. By 1.
3. The claim holds by finite arithmetic if $|s| < \omega$, $\kappa_i < \omega$ for all $i \in s$.
The claim holds by cardinal addition (Lemma 2.2.6) and cardinal multiplication (Lemma 2.2.10), if $|s| < \omega$ and $\kappa_i \geq \omega$ for some $i \in s$.
Suppose that $|s| \geq \omega$. We use 1.
Let
$$f : s \to \prod_{i\in s}\kappa_i, \qquad f(i)(j) = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}, \text{ for } i,j \in s.$$
Since f is injective, $|s| \leq \prod_{i\in s}\kappa_i$.
We have $\kappa \leq \prod_{i\in s}\kappa_i$, because $\kappa_i \leq \prod_{j\in s}\kappa_j$.

□

Lemma 2.3.4. *Suppose that $\kappa_i, \lambda_i \in \mathrm{Card}$ with $\kappa_i < \lambda_i$ for all $i \in s$ and $\kappa = \sup_{i\in s}\kappa_i$.*

1. *$\prod_{i\in s}\kappa_i \leq \kappa^{|s|} = \left|{}^{|s|}\kappa\right|$.*
2. *If $s = \mu \in \mathrm{Card}\setminus\omega$ and $0 < \kappa_i \leq \kappa_j$ for all $i < j < \mu$, then $\prod_{i\in s}\kappa_i = \kappa^{|s|}$.*
3. ***König's Theorem:***
$$\sum_{i\in s}\kappa_i < \prod_{i\in s}\lambda_i.$$

Proof.
1. 1. is clear.
2. Since $\mu\cdot\mu = \mu$, we partition μ into μ-many disjoint sets A_α for $\alpha < \mu$ of size μ.
Then
$$\prod_{i\in s}\kappa_i = \prod_{\alpha<\mu}\left(\prod_{i\in A_\alpha}\kappa_i\right).$$

Since A_α is unbounded in μ for every $\alpha < \mu$, $\prod_{i \in A_\alpha} \kappa_i \geq \sup_{i \in A_\alpha} \kappa_i = \kappa$.
So $\prod_{i \in \mu} \kappa_i \geq \kappa^\mu$.

3. Suppose that $|X_i| = \lambda_i$ for each $i \in s$ and $X = \prod_{i \in s} X_i$.
Then $|X| = \prod_{i \in s} \lambda_i$.
Suppose that $\prod_{i \in s} \lambda_i \leq \sum_{i \in s} \kappa_i$.
Find $A_i \subseteq X$ with $|A_i| \leq \kappa_i$ for each $i \in s$ such that X is the disjoint union of the sets A_i.
Let $B_i = \{f(i) \mid f \in A_i\} \subseteq X_i$ for $i \in s$, the ith projection of A_i.
Then $|X_i| = \lambda_i > \kappa_i \geq |A_i| \geq |B_i|$.
Find $x_i \in X_i \setminus B_i$ for all $i \in s$, using the Axiom of Choice.
Then $(x_i)_{i \in s} \in X$. Suppose that $f = (x_i)_{i \in s} \in A_j$ for some $j \in s$.
Then $x_j \in B_j$, contradiction.

□

2.4 Cofinality

The *cofinality* of an ordinal describes how well the ordinal can be approached from below. For example, the cofinality of $\aleph_\omega$ is ω, since the sequence $(\aleph_n)_{n \in \omega}$ has length ω and supremum $\aleph_\omega$. We will also see that the cofinality of $\aleph_n$ is $\aleph_n$ for all n, i.e. there is no sequence shorter than $\aleph_n$ with supremum $\aleph_n$.

Definition 2.4.1. Suppose that γ is a limit ordinal.

1. A set $x \subseteq \gamma$ is called *cofinal* or *unbounded* in γ if $\sup x = \gamma$.

2. A function $f : \alpha \to \gamma$ is called *cofinal* if $\mathrm{range}(f)$ is cofinal in γ.

3. The cofinality $\mathrm{cof}(\gamma)$ of γ is the least ordinal α such that there is a cofinal function $f : \alpha \to \gamma$.

Lemma 2.4.2. *Suppose that γ is a limit ordinal.*

1. $\mathrm{cof}(\gamma) \leq \gamma$.

2. $\mathrm{cof}(\gamma)$ *is always an infinite cardinal.*

3. *There is always a strictly monotone continuous cofinal function $f : \mathrm{cof}(\gamma) \to \gamma$.*

4. $\mathrm{cof}(\gamma)$ *is the least* $\mathrm{type}(x, \in)$, *where $x \subseteq \gamma$ is unbounded.*

5. $\mathrm{cof}(\gamma)$ *is the least* $|x|$, *where $x \subseteq \gamma$ is unbounded.*

6. $\mathrm{cof}(\mathrm{cof}(\gamma)) = \mathrm{cof}(\gamma)$.

Proof. Lecture 9 5th Nov

1. Note that $\mathrm{id}_\gamma : \gamma \to \gamma$ is a cofinal function from $\gamma \to \gamma$. Since α is the least ordinal with the property that such a function exists, α is at most γ.

2. $\operatorname{cof}(\omega)$ is an infinite ordinal, i.e. $\operatorname{cof}(\gamma) \geq \omega$, because any function with finite range cannot be cofinal in a limit ordinal, because functions with finite domains are always bounded.
   Suppose that $\operatorname{cof}(\gamma) \notin \mathrm{Card}$. Then, for some $\beta < \operatorname{cof}(\gamma)$, there is a surjection $g : \beta \to \operatorname{cof}(\gamma)$.
   Suppose that $f : \operatorname{cof}(\gamma) \to \gamma$ is cofinal.
   Then, $f \circ g : \beta \to \gamma$ is cofinal, because $\operatorname{range}(g) = \operatorname{dom}(f)$, thus $\operatorname{range}(f \circ g) = \operatorname{range}(f)$, contradicting the minimality of $\operatorname{cof}(\beta)$. Thus, $\operatorname{cof}(\gamma) \in \mathrm{Card}$.

3. Suppose that $f : \operatorname{cof}(\gamma) \to \gamma$ is cofinal. We define a strictly monotone continuous cofinal function $g : \operatorname{cof}(\gamma) \to \gamma$ in terms of f and by recursion for $\alpha \subseteq \operatorname{cof}(\gamma)$ as follows:
$$g(\alpha) = \begin{cases} f(0), & \text{if } \alpha = 0 \\ \max\{g(\beta)+1, f(\alpha)\}, & \text{if } \alpha = \beta+1 \\ \sup_{\beta<\alpha} g(\beta), & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$
   To check that this is well-defined, note that $\max\{g(\beta)+1, f(\alpha)\} < \gamma$ if $\alpha < \gamma$. Moreover, if $\alpha < \operatorname{cof}(\gamma)$, and $g(\beta) < \gamma$ for all $\beta < \alpha$, then $\sup_{\beta<\alpha} g(\beta) < \gamma$, by definition of $\operatorname{cof}(\gamma)$.

4. If $x \subseteq \gamma$ is cofinal (or, synonymously, unbounded) and $\operatorname{type}(x, \in) = \alpha$, then the order preserving enumeration $f : \alpha \to x$ is cofinal. So $\operatorname{cof}(\gamma) \leq \operatorname{type}(x, \in)$.
   If $\alpha = \operatorname{cof}(\gamma)$, then by 3. there is a strictly monotone continuous cofinal $g : \alpha \to \gamma$. Let $x = \operatorname{range}(g)$. Then $\operatorname{type}(x, \in) = \operatorname{cof}(\gamma)$.

5. If $x \subseteq \gamma$ is unbounded, then $\operatorname{type}(x, \in) \geq \operatorname{cof}(gamma)$, by 4.
   Then $|x| \geq \operatorname{cof}(\gamma)$, since $\operatorname{cof}(\gamma) \in \mathrm{Card}$, by 2.

6. Suppose that $f : \operatorname{cof}(\gamma) \to \gamma$ is cofinal and strictly monotone (by 3.) and $g : \operatorname{cof}(\operatorname{cof}(\gamma)) \to \operatorname{cof}(\gamma)$ is cofinal and strictly monotone (by 3.).
   We claim that $f \circ g : \operatorname{cof}(\operatorname{cof}(\gamma)) \to \gamma$ is cofinal.
   If $\beta < \gamma$, find $\alpha \in \operatorname{cof}(\gamma)$ with $f(\alpha) \geq \beta$ (since f is cofinal and $\operatorname{range}(f)$ thus unbounded, the desired α exists). Analogously, find $\alpha' \in \operatorname{cof}(\operatorname{cof}(\gamma))$ with $g(\alpha') \geq \alpha$.
   Then $(f \circ g)(\alpha') = f(g(\alpha')) \geq f(\alpha) \geq \beta$ and $f \circ g$ is thus cofinal in γ.

□

Lemma 2.4.3. *Suppose that λ is an infinite cardinal. Then*
$$\operatorname{cof}(\lambda) = \min\left\{\alpha \mid \exists (\kappa_i)_{i<\alpha} \in {}^{\alpha}\lambda \sum_{i<\alpha} \kappa_i = \lambda\right\}.$$

Proof. "$\leq$": Let α be minimal such that there is a sequence $(\kappa_i)_{i<\alpha}$ with $\sum_{i<\alpha} \kappa_i = \lambda$. We can assume that $\alpha < \lambda$.

Then,

$$\sum_{i<\alpha} \kappa_i = |\alpha| \cdot \sup_{i<\alpha} \kappa_i = \max\{|\alpha|, \sup_{i<\alpha} \kappa_i\},$$

by Lemma 2.3.3. Then $\sup_{i<\alpha} \kappa_i = \lambda$.

"$\geq$": Suppose that $f : \mathrm{cof}(\lambda) \to \lambda$ is cofinal.
Then,

$$\sum_{i<\mathrm{cof}(\lambda)} f(i) \geq \sup_{i<\mathrm{cof}(\lambda)} f(i) = \lambda.$$

□

Lemma 2.4.4. *Suppose that κ is an infinite cardinal.*
Then the set of all cofinal functions $f : \mathrm{cof}(\kappa) \to \kappa$ has size $\kappa^{\mathrm{cof}(\kappa)}$.

Proof. Exercise. □

Lemma 2.4.5. *Suppose that κ is a infinite cardinal. Then*

1. $\kappa^{\mathrm{cof}(\kappa)} > \kappa$.
2. $\mathrm{cof}(2^\kappa) > \kappa$.

Proof. 1. There is a sequence $(\kappa_i)_{i<\mathrm{cof}(\kappa)}, \kappa_i < \kappa$ with $\sum_{i<\mathrm{cof}(\kappa)} \kappa_i = \kappa$. Then, by König's Theorem (Lemma 3 (3)),

$$\kappa = \sum_{i<\mathrm{cof}(\kappa)} \kappa_i < \prod_{i<\mathrm{cof}(\kappa)} \kappa = \kappa^{\mathrm{cof}(\kappa)}.$$

2. Suppose that $\mathrm{cof}(2^\kappa) \leq \kappa$. There is a sequence $(\kappa_i)_{i<\mathrm{cof}(2^\kappa)}$ with $\kappa_i < 2^\kappa$ and $\sum_{i<\mathrm{cof}(2^\kappa)} \kappa_i = 2^\kappa$. Then,

$$2^\kappa = \sum_{i<\mathrm{cof}(2^\kappa)} \kappa_i < \prod_{i<\mathrm{cof}(2^\kappa)} 2^\kappa = (2^\kappa)^{\mathrm{cof}(2^\kappa)} = 2^{\kappa\cdot\mathrm{cof}(2^\kappa)} = 2^\kappa,$$

by König's Theorem (Lemma 3 (3)) and the assumption that $\mathrm{cof}(2^\kappa) \leq \kappa$, contradiction.

□

Definition 2.4.6. Suppose that κ is an infnite cardinal.

1. κ is called *regular* if $\mathrm{cof}(\kappa) = \kappa$.
2. κ is called *singular* if $\mathrm{cof}(\kappa) < \kappa$.

Lemma 2.4.7. *1. If κ is an infinite cardinal, κ^+ is regular.*

2. $\aleph_\omega$ is the least singular cardinal.

Proof. 1. There is a sequence $(S_\alpha)_{\alpha<\mathrm{cof}(\kappa^+)}$ within

   (i) $S_\alpha \subseteq \kappa^+$.

   (ii) $|S_\alpha| < \kappa^+$.

   (iii) $\kappa^+ = \bigcup_{\alpha<\mathrm{cof}(\kappa^+)} S_\alpha$.

   If $\mathrm{cof}(\kappa^+) < \kappa^+$, then $\mathrm{cof}(\kappa^+) \leq \kappa$. Then,

$$\left| \bigcup_{\alpha<\mathrm{cof}(\kappa^+)} S_\alpha \right| \leq \kappa \cdot \kappa = \kappa,$$

   contradiction.

2. By 1.

□

2.5 Cardinal Exponentiation

The values of cardinal exponentiation for infinite cardinals are not decided by ZFC, for example it is not provable in ZFC that $2^{\aleph_0} = \aleph_1$ and it is not provable that $2^{\aleph_0} \neq \aleph_1$.

In this section, we will see how the *continuum function* mapping κ to 2^κ is related to the *Gimel function* mapping κ to $\kappa^{\mathrm{cof}(\kappa)}$.

Definition 2.5.1. 1. The *continuum hypothesis* (CH) is the statement $2^\omega = \omega_1$.

2. The *generalised continuum hypothesis* (GCH) is the statement:

$$\forall \kappa \in \mathrm{Card} \setminus \omega \; 2^\kappa = \kappa^+.$$

3. A cardinal κ is a *strong limit cardinal* if

$$\forall \mu \in \mathrm{Card} \cap \kappa \; 2^\mu < \kappa.$$

$$\aleph_0 \quad \aleph_1 = 2^{\aleph_0} \quad \aleph_2 = 2^{\aleph_1} \quad \cdots \quad \aleph_{n+1} = 2^{\aleph_n} \quad \cdots \quad \aleph_\omega \longrightarrow \mathrm{Card}, \mathsf{GCH}$$

Definition 2.5.2. Suppose that κ, λ are infinite cardinals.

1. $\kappa^{<\lambda} := \sup_{\alpha<\lambda} \kappa^{|\alpha|}$.

2. $(<\kappa)^\lambda := \sup_{\alpha<\kappa} |\alpha|^\lambda$.

3. ${}^{<\lambda}x = \bigcup_{\alpha<\lambda} {}^\alpha x$.

Remark 2.5.3. The function $f : \mathrm{Card} \setminus \omega \to \mathrm{Card} \setminus \omega$, $f(\kappa) = 2^\kappa$, satisfies the following properties for all $\kappa, \lambda \in \mathrm{Card} \setminus \omega$:

1. $\kappa \le \lambda \Rightarrow 2^\kappa \le 2^\lambda$.
2. $\kappa < \mathrm{cof}(2^\kappa) \le 2^\kappa$.

For regular cardinals κ, λ, these are the only properties of f provable in ZFC, by Easton's Theorem [3, 5].

Remark 2.5.4. Lecture 10 10th Nov

1. The continuum hypothesis CH ($2^{\aleph_0} = \aleph_1$) is not decided by ZFC. It is consistent with ZFC that $2^{\aleph_0} = \aleph_1$, or $2^{\aleph_0} = \aleph_2$, ..., $2^{\aleph_0} = \aleph_{n+1}$, ..., $2^{\aleph_0} = \aleph_{\omega+1}$, ...
2. The generalised continuum hypothesis GCH ($\forall \kappa \in \mathrm{Card} \setminus \omega \, 2^\kappa = \kappa^+$) is not decided by ZFC.

The next result shows how to determine the *continuum function* which maps $\kappa \in \mathrm{Card} \setminus \omega$ to 2^κ from the $\gimel$-function("*gimel*"-function) $\gimel : \kappa \mapsto \kappa^{\mathrm{cof}(\kappa)}$ for κ an infinite cardinal.

Lemma 2.5.5. *Suppose that κ, λ are infinite cardinals.*

1. *If $\kappa \le \lambda$, then $\kappa^\lambda = 2^\lambda$.*
   In particular, if κ is regular, then $2^\kappa = \kappa^\kappa = \gimel(\kappa)$.
2. $2^\kappa = (2^{<\kappa})^{\mathrm{cof}(\kappa)}$.
3. *If κ is a limit cardinal and there is no $\gamma < \kappa$ with $2^\gamma = 2^{<\kappa}$, then*
$$2^\kappa = \mu^{\mathrm{cof}(\mu)}, \text{ where } \mu = 2^{<\kappa} = \sup_{\mu \in \mathrm{Card} \cap \kappa} 2^\mu.$$
4. *If κ is a singular limit cardinal and there is some $\gamma < \kappa$ with $2^\gamma = 2^{<\kappa}$, then $2^\kappa = 2^{<\kappa}$.*

Proof.
1. $\kappa^\lambda \le (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda \le \kappa^\lambda$.
2. Suppose that $\kappa_i < \kappa$ for $i < \mathrm{cof}(\kappa)$ and $\kappa = \sum_{i<\mathrm{cof}(\kappa)} \kappa_i$. Then
$$2^\kappa = 2^{\sum_{i<\mathrm{cof}(\kappa)} \kappa_i} = \prod_{i<\mathrm{cof}(\kappa)} 2^{\kappa_i} \le \prod_{i<\mathrm{cof}(\kappa)} 2^{<\kappa} = \left(2^{<\kappa}\right)^{\mathrm{cof}(\kappa)}$$
$$\left(2^{<\kappa}\right)^{\mathrm{cof}(\kappa)} \le (2^\kappa)^{\mathrm{cof}(\kappa)} = 2^{\kappa \cdot \mathrm{cof}(\kappa)} = 2^\kappa.$$
3. There is a strictly increasing sequence $(\gamma_i)_{i<\mathrm{cof}(\kappa)}$ such that $2^{\gamma_i} < 2^{\gamma_j}$ for all $i < j < \mathrm{cof}(\kappa)$.
   Then $2^{<\kappa} = \sup_{i<\mathrm{cof}(\kappa)} 2^{\gamma_i}$, so $\mathrm{cof}\,(2^{<\kappa}) = \mathrm{cof}(\kappa)$.
   By 2., $2^\kappa = (2^{<\kappa})^{\mathrm{cof}(\kappa)} = (2^{<\kappa})^{\mathrm{cof}(2^{<\kappa})}$.

4. We can choose γ with $\gamma \geq \operatorname{cof}(\kappa)$.
   Then, by 2.,
$$2^\kappa = \left(2^{<\kappa}\right)^{\operatorname{cof}(\kappa)} = 2^{\gamma \cdot \operatorname{cof}(\kappa)} = 2^\gamma = 2^{<\kappa}.$$

$\square$

Then next two results show how to determine the cardinal exponentiation function mapping $(\kappa, \lambda) \in (\mathrm{Card} \setminus \omega)^2$ to λ^κ from the $\gimel$-function $\gimel(\kappa) = \kappa^{\operatorname{cof}(\kappa)}$ for κ an infinite cardinal.

Lemma 2.5.6. *Suppose* $\kappa, \lambda \in \mathrm{Card} \setminus \omega$.

1. *If* $\kappa < \operatorname{cof}(\lambda)$*, then*
$$\lambda^\kappa = \sum_{\alpha < \kappa} |\alpha|^\kappa = \lambda \cdot (< \lambda)^\kappa = \lambda \cdot \sup_{\mu \in \mathrm{Card} \cap \lambda} \mu^\kappa.$$

2. *If* λ *is a limit cardinal and* $\operatorname{cof}(\lambda) \leq \kappa$*, then* $\lambda^\kappa = ((< \lambda)^\kappa)^{\operatorname{cof}(\lambda)}$.

3. ***Hausdorff's formula:***
$$\left(\lambda^+\right)^\kappa = \lambda^\kappa \cdot \lambda^+.$$

Proof. 1. Since $\kappa < \operatorname{cof}(\lambda)$, every function $f : \kappa \to \lambda$ has bounded range in λ. Hence ${}^\kappa\lambda = \bigcup_{\alpha<\lambda} {}^\kappa\alpha$.
   Thus
$$\lambda^\kappa = \left| \bigcup_{\alpha < \lambda} {}^\kappa\alpha \right| \leq \sum_{\alpha < \lambda} |\alpha|^\kappa = \lambda \cdot \sup_{\alpha < \lambda} |\alpha|^\kappa = \lambda \cdot (< \lambda)^\kappa \leq \lambda^\kappa.$$

2. Suppose that $\lambda = \sum_{i < \operatorname{cof}(\lambda)} \lambda_i$ with $2 \leq \lambda_i < \lambda$ for all $i < \operatorname{cof}(\lambda)$.
$$\lambda^\kappa = \left(\sum_{i < \operatorname{cof}(\lambda)} \lambda_i \right)^\kappa \leq \left(\prod_{i < \operatorname{cof}(\lambda)} \lambda_i \right)^\kappa = \prod_{i < \operatorname{cof}(\lambda)} (\lambda_i^\kappa) \leq \prod_{i < \operatorname{cof}(\lambda)} (< \lambda)^\kappa = ((< \lambda)^\kappa)^{\operatorname{cof}(\lambda)}$$
$$((< \lambda)^\kappa)^{\operatorname{cof}(\lambda)} \leq (\lambda^\kappa)^{\operatorname{cof}(\lambda)} = \lambda^{\kappa \cdot \operatorname{cof}(\lambda)} = \lambda^\kappa.$$

3. If $\kappa < \lambda^+ = \operatorname{cof}(\lambda^+)$, then
$$\left(\lambda^+\right)^\kappa = \lambda^+ \cdot \left(< \lambda^+\right)^\kappa = \lambda^+ \cdot \lambda^\kappa.$$
   If $\kappa \geq \lambda^+$, then $(\lambda^+)^\kappa = 2^\kappa$, by Lemma 2.5.5 (1) and also $\lambda^\kappa = 2^\kappa$, so $\lambda^\kappa = 2^\kappa > \kappa \geq \lambda^+$. Hence $\lambda^\kappa \cdot \lambda^+ = \lambda^\kappa = 2^\kappa = (\lambda^+)^\kappa$.

$\square$

Lemma 2.5.7. *Suppose that* κ, λ *are infinite cardinals.*

1. *If There is some* $\mu < \lambda$ *with* $\lambda \leq \mu^\kappa$*, then* $\lambda^\kappa = \mu^\kappa$.

2. *If $\kappa < \operatorname{cof}(\lambda)$ and $\mu^\kappa < \lambda$ for all cardinals $\mu < \lambda$, then $\lambda^\kappa = \lambda$.*

3. *If $\operatorname{cof}(\lambda) \leq \kappa < \lambda$ and $\mu^\kappa < \lambda$ for all cardinals $\mu < \lambda$, then $\lambda^\kappa = \gimel(\lambda)$.*

Proof.

1. $\mu^\kappa \leq \lambda^\kappa \leq (\mu^\kappa)^\kappa = \mu^{\kappa \cdot \kappa} = \mu^\kappa$.

2. If $\lambda = \mu^+$, then by Hausdorff's formula (Lemma 2.5.6 (3)),

$$\lambda^\kappa = \left(\mu^+\right)^\kappa = \mu^\kappa \cdot \mu^+ = \mu^\kappa.$$

   If λ is a limit cardinal, then $\lambda = (<\lambda)^\kappa = \sup_{\mu \in \mathrm{Card} \cap \lambda} \mu^\kappa = \lambda$ by the assumption. Then $\lambda \leq \lambda^\kappa = \lambda \cdot (<\lambda)^\kappa = \lambda \cdot \lambda = \lambda$.

3. Since $\operatorname{cof}(\lambda) \leq \kappa < \lambda$, λ is a limit cardinal. Again: $\lambda = (<\lambda)^\kappa$.
   By Lemma 2.5.6 (2), $\lambda^\kappa = ((<\lambda)^\kappa)^{\operatorname{cof}(\lambda)} = \gimel(\lambda)$.

□

Definition 2.5.8. An uncountable cardinal κ is called *inaccessible* if κ is a regular strong limit cardinal, i.e. $\operatorname{cof}(\kappa) = \kappa$ and for all cardinals $\mu < \kappa$, $2^\mu < \kappa$.

Lemma 2.5.9. *If κ is inaccessible, then there is a singular strong limit cardinal smaller than κ.*

Proof. Consider the $\beth$-numbers ("*bet*"-numbers):

$$\begin{aligned} \beth_0 &= \aleph_0 = \omega. \\ \beth_{\alpha+1} &= 2^{\beth_\alpha}, \text{ for } \alpha \in \mathrm{Ord}. \\ \beth_\beta &= \sup_{\alpha<\beta} \beth_\alpha, \text{ for } \beta \text{ limit ordinal.} \end{aligned}$$

Then $\beth_\omega$ is singular, since $\operatorname{cof}(\beth_\omega) = \omega$. Then $\beth_\omega < \kappa$, since κ is inaccessible. □

Definition 2.5.10. If κ is a regular cardinal, let

$$H_\kappa = \{x \mid |\operatorname{tc}(x)| < \kappa\}.$$

H_κ is set set of sets with *hereditary size* $< \kappa$.

Lemma 2.5.11. *Suppose that κ is an infinite regular cardinal.*

1. *H_κ is transitive.*

2. *$H_\kappa \subseteq V_\kappa$.*

3. *If κ is uncountable, then $(H_\kappa, \in)$ is a model of ZF^-.*

4. *If κ is inaccessible, then $H_\kappa = V_\kappa$.*

5. *If κ is inaccessible, then V_κ is a model of ZFC.*

6. *If κ is inaccessible, then $|H_\kappa| = |V_\kappa| = \kappa$.*

7. *If $\kappa = \mu^+, \mu \in \mathrm{Card}$, then $|H_\kappa| = 2^\mu$.*

Proof.

Lecture 11
13th Nov

1. Suppose $x \in H_\kappa, y \in x$. Then $|\operatorname{tc}(x)| < \kappa$ and $y \in \operatorname{tc}(x)$, so $\operatorname{tc}(y) \subseteq \operatorname{tc}(x)$.
Then $|\operatorname{tc}(y)| \leq |\operatorname{tc}(x)| < \kappa$, so $y \in H_\kappa$.

2. We claim that $\operatorname{rank}[x] := \{\operatorname{rank}(y) | y \in x\} \in \mathrm{Ord}$ if x is transitive.
Suppose that x is transitive and $\operatorname{rank}[x]$ is not an ordinal.
Let $\gamma \in \operatorname{rank}[x]$ be least such that for some $\alpha \in \gamma$, $\alpha \notin \operatorname{rank}[x]$.
Find $y \in x$ with $\operatorname{rank}(y) = \gamma$.
Then $\operatorname{rank}(y) = \sup\{\operatorname{rank}(z) + 1 | z \in y\}$ $(= \gamma = \beta + 1)$. If $\gamma = \beta + 1$, there is some $z \in y$ with rank β. Then $z \in x$, since x is transitive, contradiction.
If γ is a limit ordinal, then for unboundedly many $\beta < \gamma$, there is some $z \in y$ with $\operatorname{rank}(y) = \beta$.
Then $\beta \in \operatorname{rank}[x]$ for some β. This contradicts the minimality of γ.
Recall $\operatorname{rank}(x) = \min\{\alpha \in \mathrm{Ord} \mid x \in V_{\alpha+1}\}$.
We claim that for every $x \in H_\kappa, \operatorname{rank}(x) < \kappa$.
Suppose that $x \in H_\kappa$. Then $|\operatorname{tc}(x)| < \kappa$.
We have $\operatorname{rank}(x) \leq \operatorname{rank}(\operatorname{tc}(x)) \leq \sup(\operatorname{rank}[\operatorname{tc}(x)]) + 1 < \kappa$. So $x \in V_\kappa$.

3. We need to check whether H_κ satisfies the axioms of ZF^-.

Set Ex: $\emptyset \in H_\kappa$.

Ext: Since H_κ is transitive.

Found: $\in$-relation is well-founded.

Pair: Since $|\operatorname{tc}(\{x, y\})| \leq |\operatorname{tc}(x) \cup \operatorname{tc}(y)| < \kappa$.

Inf: Since $\kappa > \omega$, so $\omega \in H_\kappa$. (Only here it is needed that κ is uncountable. It can be checked that H_ω, the set of hereditary finite sets, is a model for $\mathsf{ZF}+\neg$Inf.)

Union: $|\operatorname{tc}(\bigcup x)| \leq |\operatorname{tc}(x)| < \kappa$.

Sep: Follows from the Replacement Axiom.

Rep: Let $x \in H_\kappa, f : \kappa \to H_\kappa$.
We have $\operatorname{tc}(\operatorname{range}(f)) = \bigcup_{\alpha \in x} \operatorname{tc}(f(\alpha)), |\operatorname{tc}(f(\alpha))| < \kappa, |x| < \kappa$.
Then $|\operatorname{tc}(\operatorname{range}(f))| < \kappa$, since κ is regular.

4. Suppose κ is inaccessible.
We prove $V_\alpha \in H_\kappa$ for all $\alpha < \kappa$ by induction.
$V_0 = \emptyset \in H_\kappa$.
If $V_\alpha \in H_\kappa$, then $|V_{\alpha+1}| = 2^{|V_\alpha|} < \kappa$, since κ is a strong limit cardinal.
If $\alpha < \kappa$ is a limit ordinal and $V_\beta \in H_\kappa$ for all $\beta < \alpha$, then $|V_\alpha| = \sup\{|V_\beta| \mid \beta < \alpha\} < \kappa$, since κ is regular.
Then $V_\alpha \subseteq H_\kappa$, since H_κ is transitive.

5. We need to check whether H_κ satisfies, in addition to ZF^- (see 3.), the Power Set Axiom and the Axiom of Choice.

   Pow: Suppose that $x \in V_\kappa = H_\kappa$.
   If $x \in V_\kappa$, then $x \subseteq V_\alpha$ for some $\alpha < \kappa$.
   Then $x \in V_{\alpha+1}$. We have $P(x) \subseteq P(V_{\alpha+1}) = V_{\alpha+2} \subseteq V_\kappa$.
   By Separation on $V_\kappa = H_\kappa$, $P(x) \in V_\kappa$.

   AC: Exercise.

6. We prove $|V_\alpha| < \kappa$ by induction on $\alpha < \kappa$. This is as in 4.
   Therefore $|V_\kappa| \leq \kappa$. Also $\kappa \subseteq V_\kappa$ (recall: $V_\alpha \cap \mathrm{Ord} = \alpha$), so $|V_\kappa| \geq \kappa$.

7. Suppose that $\kappa = \mu^+, \mu \in \mathrm{Card}, |H_\kappa| = 2^\mu$.
   Since $P(\mu) \subseteq H_\kappa, 2^\mu \leq |H_\kappa|$. We claim that there is a surjection $f : P(\mu \times \mu) \to H_\kappa$.
   This implies that $2^\mu = |P(\mu \times \mu)| \geq |H_\kappa|$.
   We define $f : P(\mu \times \mu) \to H_\kappa$ as follows:
   If $x \subseteq \mu \times \mu$ is a well-founded extensional (binary) relation on μ (i.e. $\mathrm{field}(x) = \mu$), let $\pi = \pi_{(\mu,x)} \colon \mu \to z$ denote the transitive collapsing map of (μ, x), and let $f(x) = \pi(0)$.
   Otherwise, let $f(x) = 0$.
   f is well-defined: Suppose that $x \in P(\mu \times \mu)$.
   By induction for $\alpha < \mu$ along $(\mu, x), \pi(\alpha) \in H_\kappa$, since $\pi(\alpha) = \{\pi(\beta) \mid \beta < \mu, (\beta, \alpha) \in x\}$.
   f is surjective: Suppose that $y \in H_\kappa$. Since $|\operatorname{tc}(y)| < \kappa$, there is a transitive set $z \in H_\kappa$ of size μ with $y \in z$.
   Let $h : \mu \to z$ be bijective with $h(0) = y$.
   Let $x = \{(\alpha, \beta) \in \mu \times \mu \mid h(\alpha) \in h(\beta)\}$.
   Then $h = \pi_{(\mu,x)}, \pi(0) = y$, and $f(x) = y$.

□

3 Applications of the Axiom of Choice

In this section, we consider some results in ZFC which cannot be proved in ZF. Several important consequences of the Axiom of Choice, for instance the well-ordering principle and Zorn's Lemma, imply the Axiom of Choice in ZF.

3.1 Various Applications

Lemma 3.1.1. *The following are equivalent:*

1. AC.
2. *Every surjective function $f \colon x \to y$ has a* left inverse, *i.e. a function $g \colon y \to x$ with $g \circ f = \mathrm{id}_y$.*

Proof. Exercise. □

Theorem 3.1.2. *The following are equivalent:*

1. AC.
2. *The Well-ordering Theorem (every set can be well-ordered, Theorem 2.1.18 (1)).*

Proof.

"1.⇒2.": See the proof of Theorem 2.1.18 (1).

"2.⇒1.": Suppose that x is a set ad $f : x \to y$ is a function. with $f(z) \neq \emptyset$ for all $z \in x$.
Let
$$z = \bigcup_{u \in x} f(u) = \bigcup \mathrm{range}(f).$$
Suppose that $(z, <_0)$ is a well-order.
Let $g\colon x \to z$, where $g(u)$ is the $<_0$-least element of $f(u)$.
Then g is a choice function for f.

□

Definition 3.1.3. Suppose that $(x, \leq)$ is a partial order, $y \subseteq x, v \in x$. Lecture 12 17th Nov

1. y is called *chain* if $(y, \trianglelefteq y)$ is a linear order.
2. v is called a *(strict) upper bound* for y if for all $u \in y$, $u \leq v$ (for all $u \in y$, $u < v$).
3. $v \in x$ is called a *maximal element* of x, if for all $u \in x$, $u \leq v$.

Theorem 3.1.4. *In* ZF *the following are equivalent.*

1. AC.
2. Zorn's Lemma*: If $(x, \leq)$ is a partial order such that every chain has an upper bound, then there is a maximal element in $(x, \leq)$.*

*Proof*1.⇒2.: Suppose that $(x, \leq)$ is a partial order such that every chain in $(x, \leq)$ has an upper bound.
Suppose that $(x, \leq)$ has no maximal elements.
There is a function
$$f : \{y \subseteq x \mid y \text{ is a chain in } (x, \leq)\} \to x$$
such that for all such y, $f(y)$ is a strict upper bound for y.
By recursion, we define for $\alpha \in \mathrm{Ord}$.
$$g(\alpha) = \begin{cases} f(g[\alpha]), & \text{if } g[\alpha] \text{ is a chain in } (x, \leq) \\ \emptyset, & \text{otherwise.} \end{cases}$$
By induction on $\alpha \in \mathrm{Ord}$, $g[\alpha]$ is a chain in $(x, \leq)$.
Therefore $g : \mathrm{Ord} \to x$ is injective, contradicting the Replacement Axiom.

2.⇒1.: Suppose that $f : x \to y$ is a function such that $f(u) \neq \emptyset$ for all $u \in x$.
A *partial choice function* is a partial function $g : \mathrm{dom}(g) \to y, \mathrm{dom}(g) \subseteq x$ such that $g(u) \in f(u)$ for all $u \in \mathrm{dom}(g) \subseteq x$.
We order the set of partial choice functions for f by inclusion.
Then for any chain $z, \bigcup z$ is an upper bound for z.
By Zorn's Lemma, there is a maximal partial choice function g'.
We show that then $\mathrm{dom}(g') = x$. Suppose $\mathrm{dom}(g') \neq x$, then there is some $u \in x \setminus \mathrm{dom}(g')$. Let $z \in f(u)$. Let $g'' = g' \cup \{(u, z)\}$, contradicting the maximality of g'.
We may choose z here, because we are in need of just one choice (the Axiom of Choice is needed only for infinitely many simultaneous choices).

□

Lemma 3.1.5. *Every vector space V over a field K has a K-basis, i.e. a maximal K-linearly independent subset of V.*

Proof. Apply Zorn's Lemma to the set of all K-linearly independent subsets of V, ordered by inclusion. □

Lemma 3.1.6.

1. *$\mathbb{R}$ as a $\mathbb{Q}$-vector space has a basis.*
2. *The $\mathbb{Q}$-vector space $\mathbb{R}$ has $2^{(2^\omega)}$ many $\mathbb{Q}$-vector space automorphisms.*

Proof.

1. By the previous lemma.
2. Suppose that $B \subseteq \mathbb{R}$ is a $\mathbb{Q}$-basis für $\mathbb{R}$.
Then $|\mathbb{R}| \leq |{}^{<\omega}\mathbb{Q}| \cdot |{}^{<\omega}B| = \max\{|\mathbb{Q}|, |B|\}$, by Problem 15.
Therefore, $|B| = 2^\omega$.
There are $2^{(2^\omega)}$ many permutations of B by Problem 15.
Each permutation of B defines a vector space automorphism of $\mathbb{R}$.
There are at most $|{}^{\mathbb{R}}\mathbb{R}| = |\mathbb{R}|^{|\mathbb{R}|} = (2^\omega)^{(2^\omega)} = 2^{(2^\omega)}$ possible such automorphism, since there are not more self-mappings of $\mathbb{R}$.

□

Definition 3.1.7. Suppose that E is an equivalence relation on a set x.
A *transversal* for E is a set $T \subseteq x$ such that

1. $\forall y \in x \exists z \in T\ (y, z) \in E$.
2. $\forall y, z \in T, y \neq z \Rightarrow \neg yEz$.

In this case, $|T| = |X/E|$.

Lemma 3.1.8. *In* ZF*, the following are equivalent.*

1. AC.
2. *For any set x and any equivalence relation E on x, there is a transversal for E.*

Proof. Exercise. ☐

Lemma 3.1.9. *Let E denote the equivalence relation on $\mathbb{R}$ defined by $(x, y) \in E$ if $x - y \in \mathbb{Q}$.*

1. *There is a transversal for E.*
2. *Any transversal T for E is not Lebesgue measurable.*

Proof.

1. By the previous lemma.
2. Let $T_p = \{p + x \mod 1 \mid x \in T\}$ for $p \in \mathbb{Q}$. Then $T_p \subseteq [0, 1)$.
   Then
   (a) $T_p \cap T_q = \emptyset$ for $p, q \in \mathbb{Q} \cap [0, 1), p \neq q$:
   If $y \in T_p \cap T_q$, then $y = p + y_0 \mod 1 = q + y_1 \mod 1$. with $y_0, y_1 \in T$, so $y_0 - y_1 \mod 1 = q - p \in \mathbb{Q}$, so $y_0 = y_1$ and $p = q$, contradiction.
   (b) $[0, 1) = \bigcup_{p\in\mathbb{Q}\cap[0,1)} T_p$, by definition of T_p:
   If $z \in [0, 1)$, then $z = p + x$ for some $x \in T, p \in \mathbb{Q}$. Find $p' \in [0, 1) \cap \mathbb{Q}$ with $p - p' \in \mathbb{Z}$. We claim that $z \in T_{p'}$. Then $z = p + x = p' + x \mod 1$. So $z \in T_{p'}$.

   We claim that T is not Lebesgue measurable.
   Let λ denote the Lebesgue measure on $\mathbb{R}$.
   We have $\lambda([0, 1)) = \sum_{p\in\mathbb{Q}\cap[0,1)} \lambda(T_p)$ by (a),(b).
   Note that for all $p, q \in \mathbb{Q} \cap [0, 1), \lambda(T_p) = \lambda(T_q)$ (because T_p and T_q are only translations of one another).
   If $\lambda(T_p) = 0$ for some (all) $p \in \mathbb{Q} \cap [0, 1)$, then $\lambda([0, 1)) = 0$, contradiction.
   If $\lambda(T_p) > 0$ for some (all) $p \in \mathbb{Q} \cap [0, 1)$, then $\lambda([0, 1)) = \infty$, contradiction.
   Thus T is not Lebesgue measurable.

☐

Definition 3.1.10.

1. If $x, y \in {}^\omega 2$ with $x \neq y$, let
$$\Delta_{x,y} = \{n \in \omega \mid x(n) \neq y(n)\}.$$
2. Let E_0 denote the following equivalence relation on ${}^\omega 2$:
   Let xE_0y if and only if $\Delta_{x,y}$ is finite.
3. A *flip set* A is a set $A \subseteq {}^\omega 2$ such that the following condition holds:
   For all $x, y \in {}^\omega 2$ with $x(i) \neq y(i)$ for exactly one $i \in \omega$, $x \in A \leftrightarrow y \notin A$.

Lemma 3.1.11. *There is a flip set.*

Proof. Let T be a transversal for E_0.
Let A denote the set of all $x \in {}^\omega 2$ such that for the unique $y \in T$ E_0-equivalent to x, $|\Delta_{x,y}|$ is even. Then A is a flip set. ☐

Lemma 3.1.12. *Every set $A \subseteq \mathbb{R}$ has a countable subset D which is dense in A, i.e. $D \cap I \neq \emptyset$ for every open set $\emptyset \neq I \subseteq A$.*

Proof. Let $(a_i, b_i)_{i\in\omega}$ eunmerate the open intervals in $\mathbb{R}$ with $a_i, b_i \in \mathbb{Q}$ and $A \cap (a_i, b_i) \neq \emptyset$. Find $x_i \in A \cap (a_i, b_i)$ and let $D = \{x_i \mid i \in \omega\}$. Then D is dense in A. □

Definition 3.1.13 (Standard topology of $\mathbb{R}$)**.** Suppose $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$.

1. $A \subseteq \mathbb{R}$ is called *open* (with respect to the standard topology of $\mathbb{R}$) if for all $a \in A$ there is an $\varepsilon > 0$ such that $B_\varepsilon(a) := \{r \in \mathbb{R} \mid -\varepsilon < r - a < \varepsilon\} \subseteq A$.

2. $A \subseteq \mathbb{R}$ is called *closed* (with respect to the standard topology of $\mathbb{R}$) if $(\setminus A) \subseteq \mathbb{R}$ is open with respect to the standard topology of $\mathbb{R}$.

3. $f : A \to \mathbb{R}$ is called *continuous* (with respect to the standard topology of $\mathbb{R}$) if for any open subset $O \subseteq \mathbb{R}$, $f^{-1}(O) \subseteq A$ open with respect to the subspace topology of $\mathbb{R}$ for A, i.e. for every open subset $O \subseteq \mathbb{R}$ there is an open subset $U \subseteq \mathbb{R}$ with $U \cap A = f^{-1}(O)$. More generally, a function $f : x \to y$ is called continuous if the preimage of every open subset of y is an open subset of x with respect to the respective topologies on x and y.

4. $A \subseteq \mathbb{R}$ is called *sequentially closed* if for any sequence $(a_i)_{i\in\omega}$ with $a_i \in A$ for every $i \in \omega$, $\lim_{i\to\infty} a_i \in A$.

5. $f : A \to\subset \mathbb{R}$ is called *sequentially continuous*, if for every sequence $(a_i)_{i\in\omega}$ with $a_i \in A$ for every $i \in \omega$ and $\lim_{i\to\infty} a_i = a$, $\lim_{i\to\infty} f(a_i) = f(a)$.

Lemma 3.1.14. *1. If $A \subseteq \mathbb{R}$ is sequentially closed, then A is closed with resect to the topology of $\mathbb{R}$.*

2. If $A \subseteq \mathbb{R}$ and $f\colon A \to \mathbb{R}$ is sequentially continuous, then f is continuous with respect to the topology of $\mathbb{R}$.

Proof. For the first claim, suppose that A is not closed, i.e. $\mathbb{R} \setminus A$ is not open. Then there is some $x \in \mathbb{R} \setminus A$ such that for every $n \in \omega$, $A \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n}) \neq \emptyset$. Find $x_n \in A \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n})$, using the Aciom if Choice. Then $x = \lim_{n\to\infty} x_n$, contradiction.

For the second claim, suppose that f is not continuous. Then there is an interval (a, b) such that $f^{-1}[(a, b)]$ is not open. As in the first part, there are $x_n \in \mathbb{R} \setminus f^{-1}[(a, b)]$ such that $x = \lim_{n\to\infty} x_n \in f^{-1}[(a, b)]$. Then $f(x) = \lim_{n\to\infty} f(x_n) \in \mathbb{R}\setminus(a, b)$, contradiction. □

The converse direction of the previous lemma is provable in ZF.

3.2 Filters and Ultrafilters

Filters formalize the notion of largeness. Examples for filters are the filter of all subsets of the unit interval $[0,1]$ with measure 1, and the filter of all *co-countable* subsets of the unit interval $[0,1]$, i.e. the sets whose complement is countable. Lecture 13 19th Nov

Definition 3.2.1. 1. A *filter on a set* a is a set $F \subseteq P(a)$ such that

(i) $a \in F, \emptyset \notin F$.

(ii) $\forall x, y \in F\ x \cap y \in F$.

(iii) $\forall x \in F \forall y \subseteq a (x \subseteq y \Rightarrow y \in F)$.

2. An *ideal on a set* a is a set $I \subseteq P(a)$ such that $P(a) \setminus I$ is a filter on a.

3. An *ultrafilter on a set* on a set a is a filter F on a such that for all $x \in P(a)$, $x \in F$ or $a \setminus x \in F$.

4. The *Frechet filter* $\mathcal{F}$ on an infinite set a is the filter

$$\mathcal{F} = \{x \subseteq a \mid a \setminus x \text{ is finite }\}.$$

Sets with finite complement are sometimes called *co-finite*.

5. A filter F on a set a is called *maximal* if there is no filter G on a with $G \subsetneqq G$.

6. A filter is called *principal* if it contains a finite set.

Lemma 3.2.2. *A filter F on a set a is maximal if and only if F is an ultrafilter on a.*

Proof. Exercise. □

Theorem 3.2.3. *1. If F is a filter on a set a, then there is an ultrafilter U on a with $F \subseteq U$.*

2. For any infinite set a, there is a non-principal ultrafilter on a.

Proof. 1. Let $S = \{G \mid G \text{ is a filter on } a \text{ with } F \subseteq G\}$, ordered by inclusion.

Claim. *For any nonempty chain $x \subseteq S, \bigcup x$ is a filter in S.*

Proof. We have $a \in \bigcup x, \emptyset \notin \bigcup x, F \subseteq \bigcup x$.
Suppose that $y, z \in \bigcup x$, then there is some $G \in x$ with $y, z \in G$. Then $y \cap z \in G \subseteq \bigcup x$.
Suppose that $y \in \bigcup x, y \subseteq z \subseteq a$. Then there is some $G \in x$ with $y \in G$. Then $z \in G \subseteq \bigcup x$. □

By Zorn's Lemma, there is a maximal filter G on a with $F \subseteq G$.
Then G is an ultrafilter by the previous lemma.

2. Apply 1. to the Frechet filter $\mathcal{F}$.

□

4 Club Sets and Stationary Sets

Closed unbounded sets (*clubs*) of an uncountable regular cardinal κ are intuitively very large, and we can define the *club filter on* κ, i.e. the filter of those sets which contain a club. Clubs appear as sets of closure points of functions on κ.

4.1 The Club Filter

The closed unbounded sets lead to the notion of *stationary subsets of* κ, i.e. those sets which have nonempty intersection with every closed unbounded subset of κ.

Definition 4.1.1. Suppose that κ is a limit ordinal with $\mathrm{cof}(\kappa) > \omega$ and $C, S \subseteq \kappa$.

1. C is *bounded* in κ if there is some $\alpha < \kappa$ with $C \subseteq \alpha$.
2. C is *unbounded* in κ if C is not bounded in κ.
3. A *limit point* of C is an ordinal $\alpha < \kappa$ such that for all $\beta < \alpha$ there is some $\gamma \in C$ with $\beta < \gamma < \alpha$.
4. C is *closed* in κ if every limit point $\alpha < \kappa$ of C is in C, or, equivalently, if $\sup(x) \in C$ for all nonempty subsets $x \subseteq C$ which are bounded in κ.
5. A *club set* C in κ is a closed unbounded subset of κ.
6. S is called *stationary* if $S \cap D \neq \emptyset$ for every club set D in κ.

Definition 4.1.2. Let Lim denote the class of all limit ordinals.

Lemma 4.1.3. *Suppose that κ is a limit ordinal with $\mathrm{cof}(\kappa) > \omega$.*

1. $\mathrm{Lim} \cap \kappa$ *is a club set in* κ.
2. $\kappa \setminus \mathrm{Lim}$ *is not a club set in* κ.
3. *If* $\alpha < \kappa$, *then* $[\alpha, \kappa) = \{\beta < \kappa \mid \alpha \leq \beta < \kappa\}$ *is a club set in* κ.
4. *If* κ *is regular and* $\alpha < \kappa$, *then* $\{\alpha \cdot_{\mathrm{Ord}} \beta \mid \beta < \kappa\}$ *is a club set in* κ.
5. *If* C, D *are club sets in* κ, $C \cap D$ *is a club set in* κ.
6. *If* $S \subseteq \kappa$ *is unbounded, then the set of limit points* $\alpha < \kappa$ *is a club set in* κ.
7. *If* κ *is regular and* $f \colon \kappa \to \kappa$ *is a function, the set of* closure points $\alpha < \kappa$ *of* f, *i.e.* $\{\alpha < \kappa \mid f[\alpha] \subseteq \alpha\}$ *forms a club set in* κ.
8. *If* $\mu < \mathrm{cof}(\kappa)$ *regular, then*

$$E^\kappa_\mu = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \mu\} \ni \mu$$

is stationary in κ.

Proof. 1.,3.,4. clear. 2.,5.,6.,7. exercise.

8. Suppose that $C \subseteq \kappa$ is a club set in κ.
   Let $f : \lambda \to C$ be the order preserving enumeration of C.
   Since C is a club, f is continuous.
   We have $\lambda \geq \operatorname{cof}(\kappa) > \mu$. Then $f(\mu) \in C$ has cofinality μ, because f is continuous, so $\mu \in C \cap E^\kappa_\mu$.

□

Lemma 4.1.4. *Suppose that $\kappa > \omega$ is regular.*
If $\gamma < \kappa$ and C_α is a club set in κ for all $\alpha < \gamma$, then $C = \bigcap_{\alpha<\gamma} C_\alpha$ is a club set in κ.

Proof. C is closed in κ.
It remains to show that C is unbounded in κ. Suppose that $\alpha < \kappa$.
Find a strictly increasing function $f : \gamma \cdot_{\mathrm{Ord}} \omega \to \kappa$ with $f(0) \geq \alpha$ and $f(\gamma \cdot_{\mathrm{Ord}} n +_{\mathrm{Ord}} \beta) \in C_\beta$ for all $n \in \omega, \beta < \gamma$.
Then $\sup(\operatorname{range}(f)) \in C = \bigcap_{\alpha<\gamma} C_\alpha$.
The regularity of κ is used in the definition of f. □

Definition 4.1.5. Suppose that γ is a limit ordinal, and $C_\alpha \subseteq \gamma$ is a club set in γ for all $\alpha < \gamma$. The *diagonal intersection* of $(C_\alpha)_{\alpha<\gamma}$ is defined as

$$\underset{\alpha<\gamma}{\Delta} C_\alpha = \{\beta < \gamma \mid \forall \alpha < \beta \; \beta \in C_\alpha\}.$$

Lemma 4.1.6. *Suppose that $\kappa > \omega$ is a regular cardinal and $C_\alpha \subseteq \kappa$ is a club set for every $\alpha < \kappa$. Then $C = \underset{\alpha<\kappa}{\Delta} C_\alpha$ is a club set in κ.*

Proof.

Claim. *C is closed in κ.*

Proof. Suppose that $\gamma < \kappa$ is a limit point of C.
For every $\beta < \gamma$ $\left(\underset{\alpha<\kappa}{\Delta} C_\alpha\right) \setminus (\beta + 1) \subseteq C_\beta$.
So γ is a limit point of C_β, so $\gamma \in C_\beta$. Then $\gamma \in \underset{\alpha<\kappa}{\Delta} C_\alpha$. □

Claim. *C is unbounded in κ.*

Proof. Suppose that $\alpha_0 < \kappa$. Let

$$\alpha_{n+1} = \min \left\{ \left(\bigcap_{i<\alpha_n} C_i \right) \setminus (\alpha_n + 1) \right\}$$

for $n \in \omega$. For the definition of α_{n+1}, the regularity of κ is needed.
Let $\alpha = \sup_{n\in\omega} \alpha_n < \kappa$.
It remains to show that $\alpha \in \underset{\beta<\kappa}{\Delta} C_\beta$. Suppose that $\beta < \alpha$.

Find $n \in \omega$ with $\beta < \alpha_n$. Then for all $i \geq n, \beta < \alpha_i$.
Then $\alpha_{i+1} \in C_\beta$ for all $i \geq n$. Since C_β is closed in κ, $\alpha = \sup_{i>n} \alpha_{i+1} \in C_\beta$.
Hence $\alpha \in \underset{\delta<\kappa}{\Delta} C_\delta$. □

□

Definition 4.1.7. Suppose that $\gamma \in \mathrm{Ord}$ and $S \subseteq \gamma$.
A function $f : S \to \gamma$ is called *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in S$ with $\alpha \neq 0$.

Definition 4.1.8. Suppose that κ is a limit ordinal of uncountable cofinality.
Then the *club filter* $\mathcal{C}_\kappa$ on κ is the set of all $X \subseteq \kappa$ such that there is a club set C in κ with $C \subseteq X$. This is in fact a filter by Lemma 4.1.3.

Lemma 4.1.9 (Fodor's Lemma). *Suppose that κ is a regular uncountable cardinal, $S \subseteq \kappa$ stationary, $f : S \to \kappa$ regressive.* *Lecture 14 24th Nov*
Then there is a stationary set $\overline{S} \subseteq S$ such that $f \restriction \overline{S}$ is constant.

Proof. Suppose that $f^{-1}[\{i\}]$ is non-stationary for all $i < \kappa$.
Find club sets C_i in κ with $C_i \cap f^{-1}[\{i\}] = \emptyset$ for all $i < \kappa$.
Then $C := \underset{i<\kappa}{\Delta} C_i$ is closed and unbounded in κ.
Suppose that $\alpha \in S \cap C$.
Then $\alpha \in C_i$ for all $i < \alpha$. Then $f(\alpha) \neq i$ for all $i < \alpha$.
Hence, $f(\alpha) \geq \alpha$, contricting that f is regressive. □

Definition 4.1.10. Suppose that F is a filter on a cardinal κ.

1. Let $F^* := \{\kappa \setminus X \mid X \in F\}$, the *dual ideal* of F.
2. Let $F^+ := \{S \subseteq \kappa \mid \forall C \in F\ C \cap S \neq \emptyset\}$, the *$F$-positive sets* of κ.
3. Suppose that $\mu \leq \kappa$ is regular. The filter F is $<\mu$*-complete* if for all $(A_i)_{i<\gamma}$ with $A_i \in F$ for $i < \gamma, \gamma < \mu$, then $\bigcap_{i<\gamma} A_i \in F$.

Consider the club filter $\mathcal{C}_\kappa$ on κ. Then $\mathcal{C}_\kappa^+$ is the set of stationary sets of κ.

4.2 Splitting Stationary Sets

Theorem 4.2.1 (Ulam). *Suppose that $\kappa = \mu^+, \mu \in \mathrm{Card}$.*
Then there are pairwise disjoint stationary sets S_i for $i < \kappa$.

Proof. Suppose that $f_\gamma : \mathrm{cof}(\gamma) \to \gamma$ is cofinal for every $\gamma \in \kappa \cap \mathrm{Lim}$.
For $i < \mu$, let
$$A_i^j = \{\gamma < \kappa \mid i < \mathrm{cof}(\gamma) = \mathrm{dom}(f_\gamma), f_\gamma(i) = j\}.$$
Let
$$A^j = \{\gamma < \kappa \mid j \in \mathrm{range}(f_\gamma)\} = \bigcup_{i<\mu} A_i^j.$$
Note that $A_i^j \cap A_i^k = \emptyset$ for $j \neq k$.

Claim. *For κ-many $j < \kappa$, A^j is stationary in κ.*

Proof. Suppose that $j_0 < \kappa$.
Let
$$h\colon \kappa \cap \mathrm{Lim} \to \kappa, h(\alpha) = \begin{cases} \min\big(\operatorname{range}(f_\alpha) \setminus (j_0+1)\big), & \text{if } \alpha > j_0. \\ 0, & \text{if } \alpha \leq j_0. \end{cases}$$
By Fodor's Lemma (Lemma 4.1.9), there is a stationary set $S \subseteq \kappa \cap \mathrm{Lim}$ and some $j > j_0$ such that $h(\alpha) = j$ for all $\alpha \in S$.
Then $S \subseteq A^j$, since for all $\alpha \in S, j \in \operatorname{range}(f_\alpha)$.
Hence, every superset of S is also stationary. □

Claim. *For some $i < \mu$ for κ-many $j < \kappa$, A_i^j is stationary.*

Proof. If $A^j = \bigcup_{i<\mu} A_i^j$ is stationary, then for some $\alpha_j < \mu$, $A_{\alpha_j}^j$ is stationary in κ, since the intersection of μ-many club sets is a club set.
Then there is some $\alpha < \mu$ so that $\alpha_j = \alpha$ for κ-many $j < \kappa$. Let $i = \alpha$. □

This proves Ulam's Theorem. □

Ulam's Theorem also holds for any $< \kappa$-complete filter F on κ, implying there are pairwise disjoint F-positive sets $S_i \in F^+$ for $i < \kappa$.

Lemma 4.2.2. *For any regular $\kappa > \omega$, there are κ-many disjoint stationary subsets of κ.*

Proof. For $\kappa = \mu^+$, this holds by the previous theorem.
For regular limit cardinals, we have $\aleph_\alpha < \kappa$ for all $\alpha < \kappa$.
Then κ is the disjoint union of the sets $E_\lambda^\mu = \{\alpha < \kappa \mid \operatorname{cof}(\alpha) = \lambda\}$, where $\lambda < \kappa$ is regular, and each E_λ^κ is stationary. □

Theorem 4.2.3 (Solovay)**.** *Suppose that $\kappa > \omega$ is regular and $S \subseteq \kappa$ is stationary. Then there are κ-many disjoint stationary subsets of S.*

Proof. Let
$$S_{\text{sing}} = \{\alpha \in S \mid \operatorname{cof}(\alpha) < \alpha\},$$
$$S_{\text{reg}} = \{\alpha \in S \mid \operatorname{cof}(\alpha) = \alpha\}.$$

Claim. *There is a stationary set $\overline{S}$ with $\overline{S} \subseteq S_{\text{sing}}$ or $\overline{S} \subseteq S_{\text{reg}}$ and C_α for $\alpha \in \overline{S}$ with*

(i) $C_\alpha \subseteq \alpha$ is closed and unbounded.

(ii) $C_\alpha \cap \overline{S} = \emptyset$.

Proof. 1. Suppose that S_{sing} is stationary in κ.
By Fodor's Lemma (Lemma 4.1.9), there is a stationary set $\overline{S} \subset S_{\text{sing}}$ and a regular $\lambda < \kappa$ sch that $\operatorname{cof}(\alpha) = \lambda$ or all $\alpha \in \overline{S}$.
For any $\alpha \in \overline{S}$, let $C_\alpha \subseteq \alpha$ club set with $\operatorname{type}(C_\alpha) = \lambda$.
Since $\operatorname{cof}(\beta) < \lambda$ for any $\beta \in C_\alpha, C_\alpha \cap \overline{S} = \emptyset$.

2. Suppose that S_{reg} is stationary in κ.
   Let $\overline{S} = \{\alpha \in S_{\text{reg}} \mid S_{\text{reg}} \cap \alpha \text{ is non-stationary in } \alpha\}$.

   Subclaim. *$\overline{S}$ is stationary in κ.*

   Proof. Suppose that $C \subseteq \kappa \setminus (\omega + 1)$ is closed and unbounded in κ.
   Let C' denote the club set of all limit points[1] of C, then $C' \subseteq C$.
   Let $\alpha = \min(S_{\text{reg}} \cap C')$. Then $C \cap \alpha$ is a club set in α.
   Then $C' \cap \alpha$ is also a club set in α, since α is regular[2].
   Then $S_{\text{reg}} \cap (C' \cap \alpha) = \emptyset$, so $S_{\text{reg}} \cap \alpha$ is non-stationary in α.
   Hence $\alpha \in \overline{S} \cap C$. □

   For $\alpha \in \overline{S}$, let $f_\alpha \colon \operatorname{cof}(\alpha) \to \alpha$ be the order preserving enumeration of C_α.
   For $i, j < \kappa$, let

$$\begin{aligned} A_i^j &= \{\alpha \in \overline{S} \mid i \in \operatorname{dom}(f_\alpha) = \operatorname{cof}(\alpha) \text{ and } f_\alpha(i) = j\}, \\ A_i^{>j} &= \{\alpha \in \overline{S} \mid i < \operatorname{cof}(\alpha), f_\alpha(i) > j\}. \end{aligned}$$

   Then $A_i^j \cap A_i^k = \emptyset$ if $j \neq k$.

□

Claim. *There is some $i < \kappa$ and κ-many $j < \kappa$ such that $A_i^{>j}$ is stationary.*

Proof. 1. Suppose that $\overline{S} \subseteq S_{\text{sing}}$ and $\operatorname{cof}(\alpha) = \lambda$ for all $\alpha \in \overline{S}$.
   If the claim fails, then for every $i < \lambda$, there is some $\beta_i < \kappa$ such that $A_i^{>\beta_i}$ is non-stationary. Then there is a club set D_i in κ with $D_i \cap A_i^{>\beta_i} = \emptyset$.
   Then for all $\alpha \in \overline{S} \cap D_i$, $f_\alpha(i) \leq \beta_i$.
   Let $\overline{\beta} = \sup_{i<\lambda} \beta_i < \kappa$ (since κ is regular). Find $\alpha \in \overline{\cap} \left(\bigcap_{i<\lambda} D_i\right) \setminus \left(\overline{\beta} + 1\right)$.
   Then for all $i < \lambda$, $f_\alpha(i) \leq \beta_i \leq \overline{\beta}$, contradicting the assumption that f_α is cofinal in α.

2. Suppose that $\overline{S} \subseteq S_{\text{reg}}$.
   If the claim fails, then for any $i < \kappa$, there is some β_i and D_i club in κ with $D_i \cap A_i^{>\beta_i} = \emptyset$.
   Then for any $\alpha \in D_i$ with $i < \operatorname{cof}(\alpha)$, $f_\alpha(i) \leq \beta_i$.
   Let D be the club set of $\beta < \kappa$ such that $\beta_i < \beta$ for all $i < \beta$, by Lemma 4.1.3 (7).
   Find $\alpha < \beta, \alpha, \beta \in \overline{S} \cap \underset{i<\kappa}{\Delta} D_i \cap D$, which is stationary in κ, since it is the intersection of the stationary set $\overline{S}$ with a club set in κ.

Subclaim. $f_\beta(\alpha) \leq \alpha$.

[1] This is a club set by Lemma 4.1.3.
[2] It would be sufficient for α to have uncountable cofinality.

Proof. It is sufficient to show that $f_\beta(i) < \alpha$ for all $i < \alpha$, since f_β is continuous and $\alpha \in \mathrm{Lim}$. Lecture 15 26th Nov
Suppose that $i < \alpha$. Since $\alpha \in D$, $\beta_i < \alpha$.
Since $i < \alpha < \beta$, $\beta \in D_i$. Hence $f_\beta(i) \leq \beta_i$, since $D_i \cap A_i^{>\beta_i} = \emptyset$.
So $f_\beta(i) \leq \beta_i < \alpha$. □

The subclaim implies that $\underbrace{f_\beta(\alpha)}_{\in C_\beta} = \alpha \in \overline{S}$, contradicting the fact that $C_\beta \cap \overline{S} = \emptyset$. □

Claim. *There is some $i < \kappa$ and κ-many $j < \kappa$ such that A_i^j is stationary.*

Proof. Suppose that $j_0 \in \kappa$. By the previous claim, $A_i^{>j_0}$ is stationary in κ.
By Fodor's Lemma (Lemma 4.1.9) applied to $h\colon A_i^{>j_0} \to \kappa, h(\alpha) = f_\alpha(i)$, there is a stationary set $S \subseteq A_i^{>j_0}$ and $j > j_0$ such that $h(\alpha) = f_\alpha(i) = j$ for all $\alpha \in S$. Hence $S \subseteq A_i^j$. □

This completes the proof of Solovay's Theorem. □

4.3 Silver's Theorem

Definition 4.3.1. Suppose that λ is a limit ordinal.

1. The functions $f, g\colon \lambda \to V$ are called *almost disjoint* if
$$\exists \alpha_0 < \lambda \, \forall \alpha \geq \alpha_0 \; f(\alpha) \neq g(\alpha).$$
2. A set $\mathcal{F} \subseteq {}^\lambda V$ of functions is called *almost disjoint* if for all $f, g \in \mathcal{F}$ with $f \neq g$, f and g are almost disjoint.

Lemma 4.3.2. *There is an almost disjoint set $\mathcal{F} \subseteq {}^\omega\omega$ of size 2^ω.*

Proof. Find $h\colon {}^{<\omega}\omega \to \omega$ injective. For $a \in {}^\omega 2$, let $f_a\colon \omega \to \omega$, $f_a(n) = h(a{\restriction}n)$.
We claim that f_a, f_b are almost disjoint for all $a, b \in {}^\omega 2$ with $a \neq b$.
Find $n \in \omega$ with $a(n) \neq b(n)$. Then for any $m \in \omega$ with $m > n$,
$$f_a(m) = h(a{\restriction}m) \neq h(b{\restriction}m) = f_b(m).$$
Let $\mathcal{F} = \{f_a \mid a \in {}^\omega 2\}$. □

Lemma 4.3.3. *Suppose that $\kappa \in \mathrm{Card}$ singular, $\lambda = \mathrm{cof}(\kappa) < \kappa, \lambda > \omega, (\kappa_\alpha)_{\alpha < \mathrm{cof}(\kappa)}$ a continuous strictly monotone cofinal sequence in κ, $\kappa_\alpha \in \mathrm{Card}$ for all $\alpha < \mathrm{cof}(\kappa)$.
Suppose that $\mu^\lambda \leq \kappa$ for all $\mu < \kappa$.
If $\mathcal{F} \subseteq \prod_{\alpha<\lambda} A_\alpha$ is almost disjoint and $|A_\alpha| \leq \kappa_\alpha$ forall $\alpha < \lambda$, then $|\mathcal{F}| \leq \kappa$.*

Proof. Assume that $A_\alpha = \kappa_\alpha$ for all $\alpha < \lambda$.

Claim. *For any $f \in \mathcal{F}$ there is a set $S_f \subseteq \lambda$ which is unbounded in λ such that* $\mathrm{range}(f{\restriction}S_f)$ *is bounded in κ.*

Proof. Let $h_f\colon \lambda \cap \mathrm{Lim} \to \lambda$ where $h_f(\alpha)$ is the least $\beta < \lambda$ with $f(\alpha) < \kappa_\beta$. Then h_f is regressive.
By Fodor's Lemma (Lemma 4.1.9), there is a stationary set $S_f \subseteq \lambda \cap \mathrm{Lim}$ such that $h_f \restriction S_f$ is constant.
Then there is some $\beta < kappa$ such that for all $\alpha \in S_f$, $f(\alpha) < \kappa_\beta$.
So range$(f \restriction S_f)$ is bounded in κ. □

Claim. *If $f, g \in \mathcal{F}$ and $f \restriction S_f = g \restriction S_g$, then $f = g$.*

Proof. Since any $f, g \in \mathcal{F}$ with $f \neq g$ are almost disjoint. □

There are at most $2^\lambda \leq \kappa$ many sets $S_f \subseteq \lambda$.
For any $S \subseteq \lambda$, the et of functions $f : S \to \kappa$ with $\sup(\mathrm{range}(f)) < \kappa$ has size at most $(<\kappa)^\lambda = \sup_{\mu \in \lambda \cap \mathrm{Card}} \mu^\lambda \leq \kappa$.
Then $|\mathcal{F}| \leq \kappa$. □

Lemma 4.3.4. *Suppose that $\kappa \in$ Card singular, $\lambda = \mathrm{cof}(\kappa) < \kappa, \lambda > \omega, (\kappa_\alpha)_{\alpha < \mathrm{cof}(\kappa)}$ a continuous strictly monotone cofinal sequence in κ, $\kappa_\alpha \in$ Card for all $\alpha < \mathrm{cof}(\kappa)$, and $\mu^\lambda \leq \kappa$ for all cardinals $\mu < \kappa$ (as in the previous lemma).
If $\mathcal{F} \subseteq \prod_{\alpha<\lambda} A_\alpha$ is almost disjoint and $|A_\alpha| \leq \kappa_\alpha^+$, then $|\mathcal{F}| \leq \kappa^+$.*

Proof. Assume that $A_\alpha = \kappa_\alpha^+$ for all $\alpha < \lambda$.

Claim. *If $g \in \mathcal{F}$ and $S \subseteq \lambda$ is unbounded,then*

$$\mathcal{F}_{g,S} = \{f \in \mathcal{F} \mid \forall \alpha \in S\ f(\alpha) \leq g(\alpha)\}$$

has size at most κ.

Proof. The map $h\colon \mathcal{F}_{g,S} \to \prod_{\alpha \in S}(f(\alpha)+1), h(f) = f \restriction S$ is injective, since $\mathcal{F}$ is almost disjoint.
Let $A_\alpha = f(\alpha)+1$ for $\alpha \in S$ and let $A_\alpha = 1$ for all $\alpha \in \kappa \setminus S$.
By the previous lemma, $|\mathcal{F}_{g,S}| \leq \kappa$. □

Let

$$f \leq_{\mathrm{ub}} g \text{ if } \{\alpha < \lambda \mid f(\alpha) \leq g(\alpha)\} \text{ is unbounded in } \lambda.$$
$$\mathcal{F}_g = \{f \in \mathcal{F} \mid f \leq_{\mathrm{ub}} g\} = \bigcup \{\mathcal{F}_{g,S} \mid S \subseteq \lambda \text{ is unbounded in } \lambda\}.$$

Claim. *For all $g \in \mathcal{F}$, $|\mathcal{F}_g| \leq \kappa$.*

Proof. $|\mathcal{F}_g| \leq 2^\lambda \cdot \kappa = \kappa$. □

Claim. *There is a sequence $(f_\alpha)_{\alpha<\delta}$ in $\mathcal{F}$ for some $\delta \leq \kappa^+$ such that*

$$\forall g \in \mathcal{F}\ \exists \alpha < \delta\ g \leq_{\mathrm{ub}} f_\alpha.$$

Then $g \in \mathcal{F}_{f_\alpha}$ and $\mathcal{F} = \bigcup_{\alpha<\delta} \mathcal{F}_{f_\alpha}$.

Proof. Define by recursion.
Let $f_0 \in \mathcal{F}$ be arbitrary.
Suppose that $\beta \in \mathrm{Ord}$ and f_α is defined for all $\alpha < \beta$.
Find some $f_\beta \in \mathcal{F}$ with $f_\beta \not\leq_{ub} f_\alpha$ for all $\alpha < \beta$, if this exists.
Otherwise, let $S = \beta$.
Then for all $\alpha < \beta < \delta$, $f_\beta \not\leq_{ub} f_\alpha$, so $f_\alpha \leq_{\mathrm{ub}} f_\beta$, and thus $f_\alpha \in \mathcal{F}_{f_\beta}$.
By the previous claim, $\left|\mathcal{F}_{f_\beta}\right| \leq \kappa$ for all $\beta < \delta$, so $\beta < \kappa^+$. Then $\delta \leq \kappa^+$. □

Then $\mathcal{F} = \bigcup_{\alpha<\delta} \mathcal{F}_{f_\alpha}$, so $|\mathcal{F}| \leq \kappa \cdot \kappa^+ = \kappa^+$. □

Theorem 4.3.5 (Silver)**.** *Suppose that κ is a singular cardinal of uncountable cofinality. If the GCH holds below κ, i.e. $2^\mu = \mu^+$ for all infinite cardinals $\mu < \kappa$, then $2^\kappa = \kappa^+$.*

Proof. Let $\lambda = \mathrm{cof}(\kappa)$ and $(\kappa_\alpha)_{\alpha<\lambda}$ be a continuous strictly monotone cofinal sequence in $\kappa \cap \mathrm{Card}$.
Let
$$f\colon {}^{\kappa}2 \to \prod_{\alpha<\lambda} {}^{\kappa_\alpha}2, f(x) = (x{\restriction}\kappa_\alpha)_{\alpha<\lambda}.$$

Claim. *1. f is injective.*

2. $\mathcal{F} = \{f(x) \mid x \in {}^{\kappa}2\}$ is an almost disjoint set.

Proof. 1. If $x(\beta) \neq y(\beta)$ and $\kappa_\alpha > \beta$, then $x{\restriction}\kappa_{\alpha'} \neq y{\restriction}\kappa_{\alpha'}$ for $\alpha' > \alpha$.

2. If $x \neq y$, then $f(x), f(y)$ are almost disjoint.

□

By GCH below κ, $2^{\kappa_\alpha} = \kappa_\alpha^+$. By the previous lemma, $|\mathcal{F}| \leq \kappa^+$.
Since f is injective, $2^\kappa = \kappa^+$. □

Remark 4.3.6. 1. It is consistent with ZFC that $\kappa > \omega$ is regular, GCH holds below κ, and $2^\kappa > \kappa^+$, by Easton's Theorem[3, 5].

2. It is consistent with ZFC that $\kappa > \omega$ is singular, $\mathrm{cof}(\kappa) = \omega$, and GCH holds below κ, but $2^\kappa > \kappa^+$, by a theorem of Magidor[8].

5 Trees and Partitions

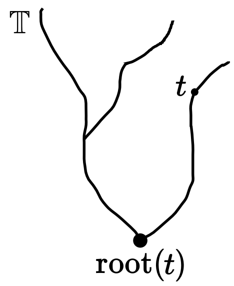
Lecture 16
1st Dec

5.1 Introduction

Remember that a partial order $(P, \leq_P)$ consists of a set P and a binary relation $\leq_P$ on P that is reflexive, antisymmetric and transitive.
Given such a partial order $(P, \leq_P)$, we also use P to denote the partial order itself and we let $<_P$ denote the induced strict ordering, i.e. $p <_P q :\Leftrightarrow (p \leq_P q \wedge p \neq q)$.
Moreover, if $A \subseteq P$, we also use $\leq_P$ to denote the induced ordering $\leq_P \upharpoonright (A \times A)$ on A.

Definition 5.1.1. A partial order $\mathbb{T} = (\mathbb{T}, <_\mathbb{T})$ is called a (set-theoretic) *tree* if it has the following properties.

1. There is a unique $<_\mathbb{T}$-minimal element $\mathrm{root}(\mathbb{T})$.
2. The set $\mathrm{pred}_\mathbb{T}(t) = \{s \in \mathbb{T} \mid s <_\mathbb{T} t\}$ is well-ordered by $<_\mathbb{T}$ for every $t \in \mathbb{T}$.



In some set theory textbooks, trees are defined as growing downwards, but in this lecture, we define them as growing upwards.

Example 5.1.2. Let $\alpha, \beta \in \mathrm{Ord}$ and let ${}^{<\alpha}\beta$ denote the set of all functions $f \colon \overline{\alpha} \to \beta$ with $\overline{\alpha} < \alpha$. Then $({}^{<\alpha}\beta, \subseteq)$ is a tree, $\mathrm{root}\big(({}^{<\alpha}\beta, \subseteq)\big) = \emptyset$.

Definition 5.1.3. Let $\mathbb{T}$ be a tree.

1. For $t \in \mathbb{T}$, we define the *length of* t by $\mathrm{lh}_\mathbb{T}(t) = \mathrm{type}(\mathrm{pred}_\mathbb{T}(t), <_\mathbb{T})$.
2. Given $\alpha \in \mathrm{Ord}$, we define the *α-th level of* $\mathbb{T}$ by $\mathbb{T}(\alpha) = \{t \in \mathbb{T} \mid \mathrm{lh}_\mathbb{T}(t) = \alpha\}$, and $\mathbb{T}_{<\alpha} = \{t \in \mathbb{T} \mid \mathrm{lh}_\mathbb{T} < \alpha\}$.
3. We define the *height of* $\mathbb{T}$ by $\mathrm{ht}(\mathbb{T}) = \min\{\alpha \in \mathrm{Ord} \mid \mathbb{T}(\alpha) = \emptyset\}$.
4. We say that $s, t \in \mathbb{T}$ are *compatible in* $\mathbb{T}$ if either $s \leq_\mathbb{T} t$ or $t \leq_\mathbb{T} s$.
   We let $\mathbb{T}_s$ denote the set of all $t \in \mathbb{T}$ that are compatible with $s \in \mathbb{T}$.
5. Given $s \in \mathbb{T}$, we say that $t \in \mathbb{T}$ is a *direct successor of s in* $\mathbb{T}$ if $s \leq_\mathbb{T} t$ and $\mathrm{lh}_\mathbb{T}(t) = \mathrm{lh}_\mathbb{T}(s) + 1$, and we let $\mathrm{succ}_\mathbb{T}(s)$ denote the set of all direct successors of s in $\mathbb{T}$.

Proposition 5.1.4. *Let $\mathbb{T}$ be a tree.*

1. *If $t \in \mathbb{T}$, then $\mathrm{lh}_{\mathbb{T}}(t) = \{\mathrm{lh}_{\mathbb{T}}(s) \mid s \in \mathrm{pred}_{\mathbb{T}}(t)\}$.*

2. *We have $\mathrm{ht}(\mathbb{T}) = \{\mathrm{lh}_{\mathbb{T}}(t) \mid t \in \mathbb{T}\}$.*

Proof. Let $(t_\alpha)_{\alpha<\mathrm{lh}_{\mathbb{T}}(t)}$ denote the monotone enumeration (i.e. the inverse of the collapsing map) of $(\mathrm{pred}_{\mathbb{T}}(t), <_{\mathbb{T}})$.
Since $\mathrm{pred}_{\mathbb{T}}(t)$ is linearly ordered by $\leq_{\mathbb{T}}$, we know that $(t_{\overline{\alpha}})_{\overline{\alpha}<\alpha}$ is the monotone enumeration of $(\mathrm{pred}_{\mathbb{T}}(t_\alpha), <_{\mathbb{T}})$ and hence $\mathrm{lh}_{\mathbb{T}}(t) = \alpha$.

If $t \in \mathbb{T}$, then $\mathbb{T}(\mathrm{lh}_{\mathbb{T}}(t)) \neq \emptyset$ and $\mathbb{T}(\alpha) \neq \emptyset$ for all $\alpha < \mathrm{lh}_{\mathbb{T}}(t)$.
This shows that $\mathrm{lh}_{\mathbb{T}}(t) < \mathrm{ht}(\mathbb{T})$ for all $t \in \mathbb{T}$.
Conversely, if $\alpha < \mathrm{ht}(\mathbb{T})$, then there is $t \in \mathbb{T}(\alpha)$ and $\mathrm{lh}_{\mathbb{T}}(t) = \alpha$. □

Proposition 5.1.5. *Let $\mathbb{T}$ be a tree and $c \subseteq \mathbb{T}$ be a subset that is linearly ordered by $\leq_{\mathbb{T}}$. Then $(c, <_{\mathbb{T}})$ is a well-order.*

Proof. Assume that there is a $<_{\mathbb{T}}$-descending sequence $(t_n)_{n<\omega}$ in c. Then $(t_{n+1})_{n<\omega}$ is a $<_{\mathbb{T}}$-descending sequence in $\mathrm{pred}_{\mathbb{T}}(t_0)$, contradiction. □

Definition 5.1.6. Let $\mathbb{T}$ be a tree.

1. A *chain* in $\mathbb{T}$ is a subset $c \subseteq \mathbb{T}$ that is linearly ordered by $\leq_{\mathbb{T}}$.
   Given such a chain c, we define the length of the chain $\mathrm{lh}_{\mathbb{T}}(c) = \mathrm{type}(c, <_{\mathbb{T}})$, and we let $(c_\alpha)_{\alpha<\mathrm{lh}_{\mathbb{T}}(c)}$ denote the monotone enumeration of $(c, <_{\mathbb{T}})$.

2. A *branch through* $\mathbb{T}$ is a chain that is $\leq_{\mathbb{T}}$-downwards-closed in $\mathbb{T}$, i.e. for any branch b through $\mathbb{T}$ and $t \in b$, $\mathrm{pred}_{\mathbb{T}}(t) \subseteq b$.
   We let $\sigma\mathbb{T}$ denote the set of all branches through $\mathbb{T}$.

3. A *maximal branch through* $\mathbb{T}$ is a branch through $\mathbb{T}$ that is not a proper subset of another branch through $\mathbb{T}$.
   We let $\partial\mathbb{T}$ denote the set of all maximal branches through $\mathbb{T}$.

4. A *cofinal branch through* $\mathbb{T}$ (cofinal is meant with respect to the height) is a branch b through $\mathbb{T}$ with $\mathrm{lh}_{\mathbb{T}}(b) = \mathrm{ht}(\mathbb{T})$.
   We let $[\mathbb{T}]$ denote the set of all cofinal branches through $\mathbb{T}$.

Many important questions in set theory can be reformulated to questions about the existence oder non-existence of cofinal branches through certain trees of infinite heights.

Proposition 5.1.7. *Let $\mathbb{T}$ be a tree.*

1. *If $t \in \mathbb{T}$, then $\mathrm{pred}_{\mathbb{T}}(t) \in \sigma\mathbb{T}$ with $\mathrm{lh}_{\mathbb{T}}(\mathrm{pred}_{\mathbb{T}}(t)) = \mathrm{lh}_{\mathbb{T}}(t)$.*
   Given $\alpha < \mathrm{lh}_{\mathbb{T}}(t)$, we write $t{\restriction}\alpha$ instead of $\mathrm{pred}_{\mathbb{T}}(t)(\alpha)$.

2. *If $b \in \sigma\mathbb{T}$ and $\alpha < \mathrm{lh}_{\mathbb{T}}(b)$, then $b \cap \mathbb{T}(\alpha) = \{b(\alpha)\}$.*

Proof. 1. This follows from the transitivity of $\leq_{\mathbb{T}}$ and the definition of $\mathrm{pred}_{\mathbb{T}}(t)$.

2. Given $\alpha < \mathrm{lh}_{\mathbb{T}}(b)$, the $\leq_{\mathbb{T}}$-downwards closure of b implies that $(b(\overline{\alpha}))_{\overline{\alpha}<\alpha}$ is the monotone enumeration of $(\mathrm{pred}_{\mathbb{T}}(b(\alpha)), <_{\mathbb{T}})$.
   Hence $\mathrm{lh}_{\mathbb{T}}(b(\alpha)) = \alpha$ for all $\alpha < \mathrm{lh}_{\mathbb{T}}(b)$.
   Since b is lineary ordered by $\leq_{\mathbb{T}}$ it follows that $b(\alpha)$ is the unique element in $b \cap \mathbb{T}(\alpha)$.

□

Proposition 5.1.8. *Let $\mathbb{T}$ be a tree and $b, c \in \sigma\mathbb{T}$.*

1. *$b \cap c \in \sigma\mathbb{T}$.*
2. *If $b \nsubseteq c$, then there is $t \in b$ with $\mathrm{pred}_{\mathbb{T}}(t) = b \cap c$.*
3. *If $b \nsubseteq c$, then there is $t \in c$ with $\mathrm{pred}_{\mathbb{T}}(t) = b$.*
4. *If $b \nsubseteq c$ and $c \nsubseteq b$, then there is $\alpha < \min(\mathrm{lh}_{\approx}(b), \mathrm{lh}_{\mathbb{T}}(c))$ such that $b(\alpha) \neq c(\alpha)$. Let $\Delta(b, c)$ denote the minimal such α, then $\Delta(b, c) = \mathrm{lh}_{\mathbb{T}}(b \cap c)$.*

Proof. 1. Being a chain and being $\leq_{\mathbb{T}}$-downwards closed is $\cap$-stable.

2. Pick t $<_{\mathbb{T}}$-minimal in $b \setminus c$. The $\leq_{\mathbb{T}}$-downwards closure of c implies that $\mathrm{pred}_{\mathbb{T}}(t) = b \cap c$.
3. This follows from 2.
4. By two applications of 2., we find $s \in b$ and $t \in c$ with $b \cap c = \mathrm{pred}_{\mathbb{T}}(s) = \mathrm{pred}_{\mathbb{T}}(t)$. If $\alpha = \mathrm{lh}_{\mathbb{T}}(s)$, then $s = b(\alpha), t = c(\alpha)$, and α is minimal with $b(\alpha) \neq c(\alpha)$.

□

Lemma 5.1.9. *Let $\mathbb{T}$ be a tree.*

1. *$[\mathbb{T}] \subseteq \partial\mathbb{T}$.*
2. ***Hausdorff Maximality Principle:** If $t \in \mathbb{T}$, then there is $b \in \partial\mathbb{T}$ with $t \in b$. In particular, $\partial\mathbb{T} \neq \emptyset$.*

Example 5.1.10. 1. Let $\mathbb{T}$ be the tree of finite strictly decreasing sequences of natural numbers ordered by inclusion. Then $\mathrm{ht}(\mathbb{T}) = \omega$. Something an the n-th level looks like $f : n \to \omega, i \mapsto n - i$.
   If the restriction to finite sequences is omitted, $\mathrm{ht}(\mathbb{T}) = \omega$, since there are no infinite decreasing chains on ω.
   There are no cofinal branches through $\mathbb{T}$. For a function f in $\mathbb{T}$, when the maximal element of $\mathrm{dom}(f)$ is sent to 0, the function is maximal.

2. If κ is an infinite cardinal, there is a tree $\mathbb{T}$ with $\mathrm{ht}(\mathbb{T}) = \kappa, [\mathbb{T}] = \emptyset, |\mathbb{T}(\alpha)| \leq \mathrm{cof}(\kappa)$ for all $\alpha < \kappa$.

Lecture 17
8th Dec

Proof of Lemma 5.1.9.

1. Pick $b \in [\mathbb{T}]$ and assume there is $c \in \sigma\mathbb{T}$ with $b \subsetneq c$.
   By Lemma 5.1.9 (3), there is $t \in c$ with $b = \mathrm{pred}_{\mathbb{T}}(t)$.
   Using Proposition 5.1.4 (2) and Propositon 5.1.7 (1), we conclude $\mathrm{ht}(\mathbb{T}) = \mathrm{lh}_{\mathbb{T}}(b) = \mathrm{lh}_{\mathbb{T}}(t) < \mathrm{ht}(\mathbb{T})$, contradiction.

2. Set $B = \{b \in \sigma\mathbb{T} \mid t \in b\}$. Then $\mathrm{pred}_{\mathbb{T}}(t) \cup \{t\} \in B$ and B is closed under taking unions of $\subseteq$-increasing sequences.
   By Zorn's Lemma (Theorem 3.1.4), B contains a $\subseteq$-maximal element b.
   Assume that there is $c \in \sigma\mathbb{T}$ with $b \subseteq c$.
   Then $c \in B$ and we have $b = c$. This shows $b \in \partial\mathbb{T}$.

$\square$

The Hausdorff Maximality principle is actually equivalent to Zorn's Lemma and thus to AC. The following classical result shows that "narrow" trees of infinite heights have cofinal branches.

Theorem 5.1.11 (Kurepa)**.** *Let $\kappa > \omega$ be a regular cardinal and $\omega \leq \mu < \kappa$ be a cardinal. If $\mathbb{T}$ is a tree of height κ with $|\mathbb{T}(\alpha)| < \mu$ for all $\alpha < \kappa$, then $[\mathbb{T}] \neq \emptyset$.*

So if you have, for example, a tree of height ω_1 with only finite levels, then ,by Kurepa's Theorem, there is a cofinal branch.

Definition 5.1.12. A tree $\mathbb{T}$ is called *extensional at limit levels* if for all $s, t \in \mathbb{T}$ with $\mathrm{lh}_{\mathbb{T}}(s) = \mathrm{lh}_{\mathbb{T}}(t) \subseteq \mathrm{Lim}$, we have

$$\mathrm{pred}_{\mathbb{T}}(s) = \mathrm{pred}_{\mathbb{T}}(t) \to s = t.$$

Lemma 5.1.13. *Let $\mathbb{T}$ be a tree.*

1. *The partial order $\sigma\mathbb{T} = (\sigma\mathbb{T}, \subseteq)$ is a tree with $\mathrm{lh}_{\sigma\mathbb{T}}(b) = \mathrm{lh}_{\mathbb{T}}(b)$ for all $b \in \sigma\mathbb{T}$.*

2. *The map*
   $$\iota_{\mathbb{T}} \colon \mathbb{T} \to \sigma\mathbb{T}; t \mapsto \mathrm{pred}_{\mathbb{T}}(t) \cup \{t\}$$
   is injective and $s <_{\mathbb{T}} t$ if and only if $\iota_{\mathbb{T}}(s) \subsetneq \iota_{\mathbb{T}}(t)$ for all $s, t \in \mathbb{T}$ and $\iota_{\mathbb{T}}[\mathbb{T}(\alpha)] = \sigma\mathbb{T}(\alpha + 1)$ for all $\alpha \in \mathrm{Ord}$.

3. *The tree $\sigma\mathbb{T}$ is extensional at limit levels.*

Proof. 1. The empty set is the unique is a $\subsetneq$-decreasing sequence $(b_n)_{n<\omega}$ in $\mathrm{pred}_{\sigma\mathbb{T}}(b)$. Given $n < \omega$, there is $t_n \in b_n \setminus b_{n+1}$. Since b_{n+1} is linearly ordered by $\leq_{\mathbb{T}}$ and b_n is $\leq_{\mathbb{T}}$-downwards closed, we know $t_{n+1} <_{\mathbb{T}} t_n$, contradiction.
Now fix $b_0, b_1 \in \mathrm{pred}_{\sigma\mathbb{T}}(b)$. By Proposition 5.1.8 (3), there are $t_0, t_1 \in b$ with $b_i = \mathrm{pred}_{\mathbb{T}}(t_i), i = 1, 2$. Since b is linearly ordered by $\leq_{\mathbb{T}}$, we have either $t_0 \leq_{\mathbb{T}} t_1$ or $t_1 \leq_{\mathbb{T}} t_0$ and this implies that either $b_0 \subseteq b_1$ or $b_1 \subseteq b_0$. This shows that $(\mathrm{pred}_{\sigma\mathbb{T}}(b), \subsetneq)$ is a well-order.

Let $(b_\alpha)_{\alpha<\mathrm{lh}_{\sigma\mathbb{T}}(b)}$ be the monotone enumeration of $(\mathrm{pred}_{\sigma\mathbb{T}}(b), \subsetneq)$.
By Proposition 5.1.8 (3), there is some $t_\alpha \in b$ with $b_\alpha = \mathrm{pred}_{\mathbb{T}}(t_\alpha)$.
Then $(t_\alpha)_{\alpha<\mathrm{lh}_{\sigma\mathbb{T}}(b)}$ is the monotone enumeration of $(b, <_{\mathbb{T}})$.
Hence $\mathrm{lh}_{\mathbb{T}}(b) = \mathrm{lh}_{\sigma\mathbb{T}}(b)$.

2. Since t is the unique maximal element of $\iota_{\mathbb{T}}(t)$, we know that $\iota_{\mathbb{T}}$ is injective and order-preserving in the above sense.
If $t \in \mathbb{T}$, then $\mathrm{lh}_{\sigma\mathbb{T}}(\iota_{\mathbb{T}}(t)) = \mathrm{lh}_{\mathbb{T}}(\iota_{\mathbb{T}}(t)) = \mathrm{lh}_{\mathbb{T}}(t)+1$ and hence $\iota_{\mathbb{T}}(t) \in \sigma\mathbb{T}(\mathrm{lh}_{\mathbb{T}}(t)+1)$.
Fix $\alpha \in \mathrm{Ord}$ and $b \in \sigma\mathbb{T}(\alpha+1)$. Since $\mathrm{lh}_{\mathbb{T}}(b) = \alpha+1$, there is $t \in b \cap \mathbb{T}(\alpha)$.
Then $b = \mathrm{pred}_{\mathbb{T}}(t) \cup \{t\} = \iota_{\mathbb{T}}(t) \in \iota_{\mathbb{T}}[\mathbb{T}(\alpha)]$.

3. Let $b_0, b_1 \in \sigma\mathbb{T}$ with $b_0 \neq b_1$ and $\mathrm{lh}_{\mathbb{T}}(b_0) = \mathrm{lh}_{\mathbb{T}}(b_1) \in \mathrm{Lim}$. By Proposition 5.1.8 (4), there is $\alpha < \mathrm{lh}_{\mathbb{T}}(b_0)$ with $b_0(\alpha) \neq b_1(\alpha)$ and hence $b_0{\restriction}(\alpha+1) \neq b_1{\restriction}(\alpha+1)$.
Since $\mathrm{lh}_{\mathbb{T}}(b_0) \in \mathrm{Lim}$, we have $\alpha+1 < \mathrm{lh}_{\mathbb{T}}(b_0) = \mathrm{lh}_{\mathbb{T}}(b_1)$.

□

Lemma 5.1.14. *Let $\mathbb{T}$ be a tree, $\overline{\mathbb{T}}$ the $\subseteq$-downwards closure of* $\mathrm{range}(<_{\mathbb{T}})$ *in $\sigma\mathbb{T}$.*

1. *The partial order $\overline{\mathbb{T}} = (\overline{\mathbb{T}}, \subseteq)$ os a tree that is extensional at limit levels with $\mathrm{lh}_{\sigma\mathbb{T}}(b) = \mathrm{lh}_{\mathbb{T}}(b)$ for all $b \in \overline{\mathbb{T}}$.*
2. *If $\mathrm{ht}(\mathbb{T}) \in \mathrm{Lim}$, then $\mathrm{ht}(\mathbb{T}) = \mathrm{ht}(\overline{\mathbb{T}})$.*
3. *If μ is a cardinal with $|\mathbb{T}(\alpha)| < \mu$ for all $\alpha \in \mathrm{Ord}$, then $|\overline{\mathbb{T}}(\alpha)| < \mu$ for all $\alpha \in \mathrm{Ord}$.*
4. *If $\mathrm{ht}(\mathbb{T}) \in \mathrm{Lim}$, then there is a bijection $b\colon [\mathbb{T}] \to [\overline{\mathbb{T}}]$.*

Proof. 1. This follows from the previous lemma and the definition of $\overline{\mathbb{T}}$.

2. Assume $\mathrm{ht}(\mathbb{T}) \in \mathrm{Lim}$. If $b \in \overline{\mathbb{T}}$, then there is a $t \in \mathbb{T}$ with $b \subseteq \iota_{\mathbb{T}}(t)$ and $\mathrm{lh}_{\mathbb{T}}(b) \leq \mathrm{lh}_{\mathbb{T}}(t)+1 < \mathrm{ht}(\mathbb{T})$. Conversely, if $t \in \mathbb{T}$, then $\iota_{\mathbb{T}}(t) \in \overline{\mathbb{T}}$ and $\mathrm{lh}_{\mathbb{T}}(t) < \mathrm{lh}_{\sigma\mathbb{T}}(\iota_{\mathbb{T}}(t)) < \mathrm{ht}(\overline{\mathbb{T}})$.

3. Assume that $|\mathbb{T}(\alpha)| < \mu$ for all $\alpha \in \mathrm{Ord}$. We have $|\overline{\mathbb{T}}(0)| = 1 = |\mathbb{T}(0)| < \mu$.
Given $\alpha \in \mathrm{Ord}$, $|\overline{\mathbb{T}}(\alpha+1)| = |\iota_{\mathbb{T}}[\mathbb{T}(\alpha)]| < \mu$.
Fix $\alpha \in \mathrm{Lim}$. Given $b \in \overline{\mathbb{T}}(\alpha)$, the definition of $\overline{\mathbb{T}}$ implies that there is $t_b \in \mathbb{T}(\alpha)$ with $b \subseteq \iota_{\mathbb{T}}(t_b)$ and hence $b = \mathrm{pred}_{\mathbb{T}}(t_b)$. This shows that the function
$$f\colon \overline{\mathbb{T}}(\alpha) \to \mathbb{T}(\alpha), \quad b \mapsto t_b$$
is injective. Hence $|\overline{\mathbb{T}}(\alpha)| \leq |\mathbb{T}(\alpha)| < \mu$.

4. Define
$$\begin{aligned} f\colon [\mathbb{T}] \to [\overline{\mathbb{T}}], & b \mapsto \{\mathrm{pred}_{\mathbb{T}}(t) \mid t \in b\}, \\ g\colon [\overline{\mathbb{T}}] \to [\mathbb{T}], & B \mapsto \bigcup B. \end{aligned}$$
Then $f \circ g = \mathrm{id}_{[\overline{\mathbb{T}}]}$ and $g \circ f = \mathrm{id}_{[\mathbb{T}]}$

□

Proof of Kurepa's Theorem (Theorem 5.1.11). Let $\kappa > \omega$ be a regular cardinal, $\omega \leq \mu < \kappa$ be a cardina and $\mathbb{T}$ be a tree of height κ with $|\mathbb{T}(\alpha)| < \mu$ for all $\alpha < \kappa$.

1. Assume that μ is regular. By the previous Lemma, we may assume that $\mathbb{T}$ is extensional at limit levels. Given $\alpha \in E^\kappa_\mu = \{\alpha < \kappa \mid \operatorname{cof}(\alpha) = \mu\}$, choose $t_\alpha \in \mathbb{T}(\alpha)$.
   Since $|\mathbb{T}(\alpha)| < \mu$ and $\mathbb{T}$ is extensional at limit levels, there is $r_0(\alpha) < \alpha$ such that $t_\alpha \restriction r_0(\alpha) \neq s \restriction r_0(\alpha)$ for all $\alpha \in E^\kappa_\mu$ and $s \in \mathbb{T}(\alpha) \setminus \{t_\alpha\}$.
   This defines a regressive function $r_0 \colon E^\kappa_\mu \to \kappa$ and, by Fodor's Lemma (Theorem 4.1.9), there is $S_0 \subseteq E^\kappa_\mu$ stationary such that $r_0 \restriction S_0$ is constant with value $\alpha_0 < \kappa$.
   Choose an enumeration $(s_\gamma)_{\gamma<\lambda}, \lambda < \mu$ of $\mathbb{T}(\alpha_0)$ and we define $r \colon S_0 \setminus \mu \to \lambda$ with $t_\alpha \restriction \alpha_0 = s_{\gamma(\alpha)}$ for all $\alpha \in S_0 \setminus \mu$.
   Then $S_0 \setminus \mu$ is stationary and r is regressive, hence there is $S \subseteq S_0 \setminus \mu$ stationary with $r \restriction S$ constant by Fodor's Lemma.
   Choose $\alpha, \beta \in S$ with $\alpha < \beta$ and assume $t_\alpha \neq t_\beta \restriction \alpha$. Then we have
   $$s_{\gamma(\alpha)} = t_\alpha \restriction \alpha_0 \neq (t_\beta \restriction \alpha) \restriction \alpha_0 = t_\beta \restriction \alpha_0 = s_{\gamma(\alpha)},$$
   contradiction. This calculation shows that $\{t \in \mathbb{T} \mid \exists \alpha \in s\, t \leq_{\mathbb{T}} t_\alpha\} \in [\mathbb{T}]$.
2. Now, assume that μ is singular. Then there is $S \subseteq \kappa$ unbounded, and an infinite regular cardinal $\overline{\mu} < \mu$ such that $|\mathbb{T}(\alpha)| < \overline{\mu}$ for all $\alpha \in S$.
   Set $\mathbb{T}' = (\bigcup\{\mathbb{T}(\alpha) \mid \alpha \in S\}, \leq_{\mathbb{T}})$. Then $\mathbb{T}'$ is a tree of height κ with $|\mathbb{T}'(\alpha)| < \overline{\mu}$ for all $\alpha \in \mathrm{Ord}$. By the first part, we have $[\mathbb{T}'] \neq \emptyset$. and this implies $[\mathbb{T}] \neq \emptyset$.

□

This result leads up to one of the fundamental concepts of contemporary set theory.

Definition 5.1.15. Let κ be an infinite cardinal.

1. A *κ-Aronszajn tree* is a tree $\mathbb{T}$ with $\operatorname{ht}(\mathbb{T}) = \kappa$, $[\mathbb{T}] = \emptyset$ and $|\mathbb{T}(\alpha)| < \kappa$ for all $\alpha < \kappa$.
2. We say that κ has the *tree property* if there are no κ-Aronszajn trees.

5.2 König's Lemma

Theorem 5.2.1 (König's Lemma)**.** *The cardinal $\omega = \aleph_0$ has the tree property.* Lecture 18 10th Dec

Proof. Let $\mathbb{T}$ be a tree with $\operatorname{ht}(\mathbb{T}) = \omega$ and $|\mathbb{T}(n)| < \omega$ for all $n < \omega$. Let I denote the set of all $s \in \mathbb{T}$ such that the set $A_s = \{t \in \mathbb{T} \mid s \leq_{\mathbb{T}} t\}$ is infinte.
Since $\operatorname{ht}(\mathbb{T}) = \omega$, we have $\operatorname{root}(\mathbb{T}) \in I$.

Claim. *If $t \in I$, the $I \cap \operatorname{succ}\mathbb{T}(t) \neq \emptyset$.*

Proof. Assume that $I \cap \operatorname{succ}\mathbb{T}(t) = \emptyset$.
Since our assumptions imply that $\operatorname{succ}\mathbb{T}(t) \subseteq \mathbb{T}(\operatorname{lh}_{\mathbb{T}}(t)+1)$ is finite and $A_t = \operatorname{succ}\mathbb{T}(t) \cup \bigcup\{A_s \mid s \in \operatorname{succ}\mathbb{T}(t)\}$, we can conclude that A_t is the finite union of finite sets and therefore finite, contradiction. □

Fix a well-order $<$ of $\mathbb{T}$. We define a sequence $(t_n)_{n<\omega}$ with $t_0 = \mathrm{root}(\mathbb{T})$ and t_{n+1} is the $<$-least element in $I \cap \mathrm{succ}\,\mathbb{T}(t_n)$.
Then $\{t_n \mid n < \omega\} \in [\mathbb{T}] \neq \emptyset$. $\square$

Remark 5.2.2. The above construction uses AC for fixing a well-order. It can be shown that König's Lemma is not entailed by ZF.

We prove an application of König's Lemma in topology.

Definition 5.2.3. A topological space X is called *noetherian* if there is no sequence $(A_n)_{n<\omega}$ of closed subsets with $A_{n+1} \subsetneq A_n$ for all $n < \omega$.

Proposition 5.2.4. *Every noetherian space is compact.*

Definition 5.2.5. A topological space is called *irreducible* if every nonempty open set is dense (i.e. the intersection of two nonempty sets is nonempty). Note that a space is not irreducible if it is the union of two proper closed subsets.

Theorem 5.2.6. *Every noetherian space is the union of finitely many closed irreducible subspaces.*

Proof. Let X be a noetherian space. We inductively define a tree $\mathbb{T}$ of height at most ω by specifying the level $\mathbb{T}(n)$ and the direct successors of nodes in $\mathbb{T}(n)$ for every $n < \omega$.
Set $\mathbb{T}(0) = \{X\}$.
Assume that we constructed $\mathbb{T}(n)$. Define

$$\mathbb{T}'(n) = \{A \in \mathbb{T}(n) \mid A \subseteq X \text{ closed and irreducible}\}.$$

Given $A \in \mathbb{T}'(n)$, we find closed proper subsets A_0 and A_1 of A with $A_0 \cup A_1 = A$. Since $A \subseteq X$ closed, $A_i \subseteq X$ closed.
We define $\mathbb{T}(n+1) = \{A_i \mid A_i \in \mathbb{T}'(n), i < 2\}$ and we set A_i to be a direct successor of A in $\mathbb{T}$.
Then $\mathbb{T}$ is a tree of height at most ω and $|\mathbb{T}(n)| < \omega$ for all $n < \omega$.
Assume that $\mathrm{ht}(\mathbb{T}) = \omega$. By König's Lemma (Theorem 5.2.1), there is $b \in [\mathbb{T}]$.
Our construction ensures $b(n)$ is a closed subset of X and $b(n+1) \subsetneq b(n)$ for all $n < \omega$, contradicting that X is a noetherian space.
Hence $\mathrm{ht}(\mathbb{T}) = n < \omega$.
Then let L denote the set of all $t \in \mathbb{T}$ with $\mathrm{succ}\,\mathbb{T}(t) = \emptyset$.
Then L is finite and our construction ensures that every $A \in L$ is a closed irreducible subspace and $X = \bigcup L$.
This yields the statement of the theorem. $\square$

5.3 Ramsey's Theorem

We discuss another application of König's Lemma.

Definition 5.3.1. Let α, β, γ, and δ be ordinals with $\beta \geq \alpha \geq \gamma > 0$ and $\delta > 0$.

1. Given a set X of ordinals, we let $[X]^\gamma$ denote the set of subsets of X with order type γ.

2. We let $\beta \to (\alpha)^\gamma_\delta$ denote the statement that for every function $c\colon [\beta]^\gamma \to \delta$ there is a set $H \in [\beta]^\alpha$ such that $c{\restriction}[H]^\gamma$ is constant.
   We also call the functions c *colourings* and H *homogeneous* for c.
   We let $\beta \nrightarrow (\alpha)^\gamma_\delta$ denote the negation of this statement.

Example 5.3.2. Let κ be an infinite cardinal with $\kappa \to (\omega)^2_2$ and $\mathcal{G} = (V, E)$ be a graph of cardinality κ, i.e. V is a set of size κ and E is a binary relation on V that is antireflexive and symmetric.
Then either $\mathcal{G}$ contains an infinite complete subgraph or an infinite independent set.

The following result shows that we only need to consider colourings of finite subsets.

Lemma 5.3.3. *If $\beta \geq \omega$, then $\beta \nrightarrow (\omega)^\omega_2$.*

Proof. Fix a well-ordering $<$ of $[\beta]^\omega$ and define $c\colon [\beta]^\omega \to 2$ to be the unique function with

$$c(A) = 0 \text{ if and only if } \forall B \in [A]^\omega \big(A \neq B \to A < B\big)$$

for all $A \in [\beta]^\omega$. Assume $H \in [\beta]^\omega$ is homogeneous for c.
Let A be the $<$-least element in $[H]^\omega$. Then $c(A) = 0$, thus $c(H) = 0$, since H is homogeneous for c.
Since H is infinite, choose a sequence $(X_n)_{n<\omega}$, $X_n[H]^\omega$ for all $n < \omega$, with $X_n \subsetneq X_{n+1}$ for all $n < \omega$. This implies that $X_{n+1} < X_n$ for all $n < \omega$, contradiction. □

Lemma 5.3.4. *Let α, β, γ, and δ be ordinals with $\beta \geq \alpha \geq \gamma > 0, \delta > 0$, and $\beta \to (\alpha)^\gamma_\delta$.*

1. *Given ordinals $\bar\alpha, \bar\beta$, and $\bar\delta$ with $\bar\beta \geq \beta, \bar\alpha \leq \alpha$ and $0 < \bar\delta \leq \delta$, we have $\bar\beta \to (\bar\alpha)^\gamma_{\bar\delta}$.*

2. *If $\delta = n + 1$ for some $0 < n < \omega$ and $\alpha \geq \omega$, then $\beta \to (\alpha)^n_\delta$.*

Proof. 1. Fix $\bar c\colon [\bar\beta]^\gamma \to \bar\delta$. Then there is some $H \in [\beta]^\alpha$ homogeneous for $\bar c{\restriction}[\beta]^\gamma$ and H is also homogeneous for $\bar c$.

2. Set $\overline{H} = H \setminus \{\min(H)\} \in [\beta]^\alpha$, since $\alpha \geq \omega$.
   We have $\overline{A} \cup \{\min(H)\} \in [H]^{n+1}$ and $\bar c(\overline{A}) = c(\overline{A} \cup \{\min(H)\})$ for al $\overline{A} \in [\overline{H}]^n$.
   This shows that $\overline{H}$ is homogeneous for $\bar c$.

□

Remark 5.3.5. Lecture 19 15th Dec

1. In the above definition we may assume that $\delta \in \mathrm{Card}$. Moreover, if $\alpha \in \mathrm{Card}$, then the least β that satisfies $\beta \to (\alpha)^\gamma_\delta$ is also a cardinal.

2. The second implication of Lemma 5.3.4 does not hold without some assumption on α. For example, we have $n+2 \to (n+2)^{n+2}_2$ for all $n < \omega$, but $n+2 \nrightarrow (n+2)^{n+1}_2$ for all $n < \omega$.

The following result connects partitions with trees.

Theorem 5.3.6 (Ramification Lemma). *Let $\kappa, \nu > 0$ be cardinals, $2 \leq n < \omega \leq \kappa$ and $c\colon [\kappa]^n \to \nu$ be a function. Then there is a tree $\mathbb{T}_c = (\kappa, \leq_c)$ with the following properties:*

1. *If $\alpha, \beta < \kappa$ with $\alpha <_c \beta$, then $\alpha < \beta$.*
2. *If $\alpha_1, \dots, \alpha_{n-1}, \beta, \gamma < \kappa$ with $\alpha_0 <_c \dots <_c \alpha_{n-1} <_c \beta <_c \gamma$, then*
$$c(\{\alpha_1, \dots, \alpha_{n-1}, \beta\}) = c(\{\alpha_1, \dots, \alpha_{n-1}, \gamma\}).$$
3. *If $\alpha < \kappa$, then $|\mathbb{T}(\alpha)| \leq \nu^{|\alpha+\omega|}$.*
4. *If $\nu < \omega$, then $|\mathbb{T}(n)| < \omega$ for all $n < \omega$.*

Proof. We inductively construct trees $(\mathbb{T}^\alpha)_{n \leq \alpha \leq \kappa}$ such that the following statements hold for all $n \leq \alpha \leq \kappa$:

(a) $\mathbb{T}^\alpha = (\alpha, \leq_\alpha)$.

(b) If $n \leq \overline{\alpha} \leq \alpha$, then $\leq_{\overline{\alpha}} = \leq_\alpha \restriction (\overline{\alpha} \times \overline{\alpha}) \subseteq \trianglelefteq \restriction (\overline{\alpha} \times \overline{\alpha})$.

(c) If $n \leq \overline{\alpha} < \alpha$, there is a unique branch $b_\alpha \in \sigma \mathbb{T}^\alpha$ with $\mathrm{lh}_{\mathbb{T}^\alpha}(b_\alpha) \geq n-1$ that is maximal with the property that
$$c(\{\alpha_1, \dots, \alpha_{n-1}, \alpha\}) = c(\{\alpha_1, \dots, \alpha_{n-1}, \beta\})$$
holds for all $\alpha_1, \dots, \alpha_{n-1}, \beta \in b_\alpha$ with $\alpha_1 <_\alpha \dots <_\alpha \alpha_{n-1} <_\alpha \beta$.

(d) We let $b_\alpha = \mathrm{pred}_{\mathbb{T}^{\alpha+1}}(\alpha)$.

Set $\mathbb{T}^n = (n, \leq)$. Now assume that $n < \alpha \leq \kappa$ and we have defined $\mathbb{T}^{\overline{\alpha}}$ for all $n \leq \overline{\alpha} < \alpha$. If $\alpha \in \mathrm{Lim}$, then
$$\mathbb{T}^\alpha = \left(\alpha, \bigcup_{n \leq \overline{\alpha} < \alpha} \leq_{\overline{\alpha}}\right)$$
satisfies the above properties. Hence, we may assume that $\alpha = \overline{\alpha} + 1$.

Claim. *There is a unique $b \in \sigma \mathbb{T}^{\overline{\alpha}}$ with $\mathrm{lh}_{\mathbb{T}^{\overline{\alpha}}}(b) \geq n-1$ that is $\subseteq$-maximal with the property that $c(\{\alpha_1, \dots, \alpha_{n-1}, \overline{\alpha}\}) = c(\{\alpha_1, \dots, \alpha_{n-1}, \beta\})$ holds for all $\alpha_1, \dots, \alpha_{n-1}, \beta \in b$ with $\alpha_1 <_{\overline{\alpha}} \dots <_{\overline{\alpha}} \alpha_{n-1} <_{\overline{\alpha}} \beta$.*

Proof. Let B denote the set of all $b \in \sigma \mathbb{T}^{\overline{\alpha}}$ with the above properties.
Then $\{0, \dots, n-2\} \in B \neq \emptyset$ and B is closed under taking unions of increasing sequences. By Zorn's Lemma, B contains a $\subseteq$-maximal element $\overline{b_0}$. Assume that $\overline{b_1}$ is also maximal in B. Then $\overline{b_0} \subsetneq \overline{b_1}$ and $\overline{b_1} \subsetneq \overline{b_0}$.
By Proposition 5.1.8 (4), we can find $\beta_i \in \overline{b_i}$ with $\mathrm{pred}_{\mathbb{T}^{\overline{\alpha}}}(\beta_i) = \overline{b_0} \cap \overline{b_1}$.

Choose $k < 2$ with $\beta_k < \beta_{1-k}$.
Choose

$$\alpha_1, \ldots, \alpha_{n-1}, \beta \in b_{\beta_k} \cup \{\beta_k\} = \mathrm{pred}_{\mathbb{T}^{\overline{\alpha}}}(\beta_k) \cup \{\beta_k\} \in \sigma\mathbb{T}^{\beta_{1-k}}$$

with $\alpha_1, <_{\beta_{1-k}} \ldots <_{\beta_{1-k}} \alpha_{n-1} <_{\beta_{1-k}} \beta$. Then

$$\begin{aligned} &c(\{\alpha_1, \ldots, \alpha_{n-1}, \beta\}) = c(\{\alpha_1, \ldots, \alpha_{n-1}, \beta_k\}) \\ =&c(\{\alpha_1, \ldots, \alpha_{n-1}, \alpha\}) = c(\{\alpha_1, \ldots, \alpha_{n-1}, \beta_{1-k}\}). \end{aligned}$$

This shows that $b_{\beta_{1-k}}$ is not $\subseteq$-maximal in $\mathbb{T}^{\beta_{1-k}}$, because

$$b_{\beta_{1-k}} = \mathrm{pred}_{\mathbb{T}^{\overline{\alpha}}}(\beta_k) \subsetneq b_{\beta_k} \cup \{\beta_k\},$$

contradiction. □

Let $b_{\overline{\alpha}}$ be the branch given by the claim and define

$$\leq_\alpha = \leq_{\overline{\alpha}} \cup \{(\beta, \overline{\alpha}) \mid \beta \in b_{\overline{\alpha}}\}.$$

Then $\mathbb{T}_c = \mathbb{T}^\kappa$ is a tree that satisfies (1) and (2).

Claim. *$\mathbb{T}_c$ is extensional at limit levels.*

Proof. Choose $\gamma_0 < \gamma_1 < \kappa$ with $b_{\gamma_0} = b_{\gamma_1}$ and $\mathrm{lh}_{\mathbb{T}_c}(\gamma_i) \in \mathrm{Lim}$.
Fix $\alpha_1, \ldots, \alpha_{n-1}, \beta \in b_{\gamma_0} \cup \{\gamma_0\}$ with $\alpha_1 <_c \ldots <_c \alpha_{n-1} <_c \beta$.
Since $\mathrm{lh}_{\mathbb{T}_c}(\gamma_0) \in \mathrm{Lim}$, there is $\overline{\beta} \in b_{\gamma_0}$ with $\alpha_{n-1} <_c \overline{\beta} \leq_c \beta$. Then $\alpha_1, \ldots, \alpha_{n-1}, \overline{\beta} \in b_{\gamma_1}$ and we get

$$\begin{aligned} &c(\{\alpha_1, \ldots, \alpha_{n-1}, \gamma_1\}) = c(\{\alpha_1, \ldots, \alpha_{n-1}, \overline{\beta}\}) \\ =&c(\{\alpha_1, \ldots, \alpha_{n-1}, \gamma_0\}) = c(\{\alpha_1, \ldots, \alpha_{n-1}, \beta\}). \end{aligned}$$

This contradicts the maximality of b_{γ_1}. □

Claim. *If $n \leq \alpha < \kappa$ and $\beta \in \mathbb{T}_c(\alpha)$, then $|\mathrm{succ}_{\mathbb{T}_c}(\beta)| \leq \nu^{|\alpha|^{n-2}}$.*

Proof. Choose $\gamma_0, \gamma_1 \in \mathrm{succ}_{\mathbb{T}_c}(\beta)$ with $\gamma_0 < \gamma_1$.
By the properties of b_β, there are $\alpha_1, \ldots, \alpha_{n-2} \in b_\beta$ with $\alpha_1 <_c \ldots <_c \alpha_{n-2} <_c \beta$ and $c(\{\alpha_1, \ldots, \alpha_{n-2}, \beta, \gamma_0\}) \neq c(\{\alpha_1, \ldots, \alpha_{n-2}, \beta, \gamma_1\})$, because otherwise we would have $b_\beta \cup \{\beta, \gamma_0\} = b_{\gamma_0} \cup \{\gamma_0\} \subsetneq b_{\gamma_1} = b_\beta \cup \{\beta\}$.
There are at most $|\alpha|^{n-2}$-many tuples with this property and we can conclude that the function

$$\iota\colon \mathrm{succ}_{\mathbb{T}_c}(\beta) \to {}^{(\alpha^{n-2})}\nu;\ \gamma \mapsto \big[(\alpha_1, \ldots, \alpha_{n-2}) \mapsto c(\{\alpha_1, \ldots, \alpha_{n-2}, \beta, \gamma\})\big]$$

is injective. □

Using these two claims, we can prove (3) by induction on $\alpha < \kappa$. If $\alpha \in \mathrm{Lim}$, then the claim shows that the function

$$\iota\colon \mathbb{T}_c(\alpha) \to \prod_{\overline{\alpha}<\alpha} \mathbb{T}_c(\overline{\alpha}), \beta \mapsto \left(b_\beta(\overline{\alpha})\right)_{\overline{\alpha}<\alpha}$$

is injective and we can conclude that

$$|\mathbb{T}_c(\alpha)| \leq \left|\prod_{\overline{\alpha}<\alpha} \mathbb{T}_c(\overline{\alpha})\right| \overset{ind\,ass}{\leq} \left(\nu^{|\alpha+\omega|}\right)^{|\alpha|} = \nu^{|\alpha+\omega|}.$$

Given $\alpha < \kappa$, we have

$$|\mathbb{T}_c(\alpha+1)| = \left|\sum_{\beta\in\mathbb{T}_c(\alpha)} \mathrm{succ}_{\mathbb{T}_c}(\beta)\right| \overset{ind\,ass}{\leq} \nu^{|\alpha+\omega|} \cdot \nu^{|\alpha|^{n-2}} = \nu^{|\alpha+\omega|}.$$

Finally, if ν is finite, then we can use the last claim to show that $\mathbb{T}_c(n)$ is finite for every $n < \omega$. □

Theorem 5.3.7 (Ramsey's Theorem)**.** *If $0 < m, n < \omega$, then $\omega \to (\omega)^n_m$.*

Theorem 5.3.8 (Finite Ramsey Theorem)**.** *If $\omega > k \geq n > 0$ and $\omega > m > 0$, then there is $k \leq r < \omega$ with $r \to (k)^n_m$.*

Example 5.3.9. Given $k > 0$, we let $R(k, k)$ denote the least natural number r satisfying $r \to (k)^2_2$ – the *Ramsey number of* k.

1. $R(3.3) = 6$.
2. $R(4, 4) = 18$.
3. $43 \leq R(5, 5) \leq 49$. The Ramsey numbers are extremely difficult to compute.

The mathematician Paul Erdős commented on the Ramsey numbers:

> Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens. [11, P. 4]

Proof of Ramsey's Theorem (Theorem 5.3.7). We prove the theorem by induction on $0 < n < \omega$. In the case $n = 1$, the statement follows from the pigeonhole principle. Lecture 20 17th Dec
Fix $0 < n < \omega$ and $c\colon [\omega]^{n+1} \to m$. Let $\mathbb{T}_c = (\omega, \leq_c)$ be the tree given by the Ramification Lemma (Theorem 5.3.6). Then clause (4) of the lemma implies that all $\mathbb{T}_c(k)$ are finite

for $k < \omega$. This implies that $\mathrm{ht}(\mathbb{T}_c) = \omega$.
In this situation, König's Lemma (Theorem 5.2.1) shows that there is some $b \in [\mathbb{T}_c]$ with

$$c(\{k_1, \ldots, k_n, l_0\}) = c(\{k_1, \ldots, k_n, l_1\})$$

for al $k_1, \ldots, k_n, l_0, l_1 \in b$ with $k_1 <_c \ldots <_c k_n <_c l_0 <_c l_1$.
Then there is a function $d \colon [\omega]^n \to m, A \mapsto c(\{b(A(0)), \ldots, b(A(n-1))\})$ and the induction hypothesis yields $\overline{H} \in [\omega]^\omega$ that is homogeneous for d.
Define $H = \{b(k) \mid k \in \overline{H}\} \in [\omega]^\omega$.
Choose $B \in [H]^{n+1}$. Then $B \subseteq b$ and there is $A \in [\overline{H}]^n$ with $b(i) = b(A(i))$ for all $i < n$.
By the properties of b, we have

$$\begin{aligned} c(B) =& c(\{B(0), \ldots, B(n-1), b(\max(A)+1)\}) = d(\{A(0), \ldots, A(n-1)\}) \\ &= d(\{\overline{H}(0), \ldots, \overline{H}(n-1)\}) = c(\{H(0), \ldots, H(n)\}). \end{aligned}$$

This shows that H is homogeneous for c. □

Proof of the Finite Ramsey Theorem (Theorem 5.3.8). Given $k \leq r < \omega$, let C_r denote the set of all functions $c \colon [r]^n \to m$ such that there is no $H \in [r]^k$ that is homogeneous for c.
Assume $C_r \neq \emptyset$ for all $k \leq r < \omega$.

Claim. *If $k \leq r_0 \leq r_1 < \omega$ and $c \in C_{r_1}$, then $c \restriction [r_0]^n \in C_{r_0}$.* □

Set $C = \{\emptyset\} \cup \bigcup_{k \leq r < \omega} C_r$ and define a binary relation $\leq$ on C by

$$C_0 \leq C_1 \text{ if and only if } C_0 = \emptyset \vee \exists k \leq r_0 \leq r_1 < \omega \big(C_1 \in C_{r_1} \wedge C_0 = C_1 \restriction [r_0]^n\big).$$

Then $\mathbb{T} = (C, \leq)$ is a tree of height ω with $\mathbb{T}(n+1) = C_{k+n}$ for all $n < \omega$.
This implies that $|\mathbb{T}(n)| < \omega$ for all $n < \omega$ and König's Lemma (Theorem 5.2.1) yields $b \in [\mathbb{T}]$. Then $c = \bigcup b \colon [\omega]^n \to m$ and Ramsey's Theorem (Theorem 5.3.7) shows that there is $H \in [\omega]^\omega$ that is homogeneous for c.
But then $\{H(0), \ldots, H(k-1)\}$ is homogeneous for $b(H(k+1)) \in C_{k+H(k)}$, contradiction.

5.4 The Suslin Problem

We are now considering ω_1-Aronszajn trees. These trees are often just called Aronszajn trees. The following characterisation of the real line motivates the definition of such trees.

Proposition 5.4.1. *Let $\mathbb{L}$ be a linear order with the following properties:*

1. *$\mathbb{L}$ is dense and has no end-points.*

2. *The **linear order topology** on $\mathbb{L}$ is **connected**, i.e. $\mathbb{L}$ is not the union of two disjoint nonempty open subsets. We also say that the topological space $\mathbb{L}$ is connected. The order topology on $(\mathbb{L}, \leq_{\mathbb{L}})$ is generated by the following base:*

- *For $a \in \mathbb{L}$, the sets $\{x \in \mathbb{L} \mid x <_{\mathbb{L}} a\}$.*
- *For $a \in \mathbb{L}$, the sets $\{x \in \mathbb{L} \mid a <_{\mathbb{L}} x\}$.*
- *For $a, b \in \mathbb{L}$ with $a <_{\mathbb{L}} b$, the sets $\{x \in \mathbb{L} \mid a <_{\mathbb{L}} x <_{\mathbb{L}} b\}$.*

This means that any open set with respect to the order topology on $\mathbb{L}$ is the union of sets of the above form.

3. *The order topology of $\mathbb{L}$ is **separable** (the topological space $\mathbb{L}$ is separable, resp.), i.e. $\mathbb{L}$ contains a countable dense subsets, i.e. a subset, that has nonempty intersection with every nonempty open set.*

Then $\mathbb{L}$ is isomorphic to $(\mathbb{R}, \leq_{\mathbb{R}})$.

Proof. Let D denote the countable dense subset of $\mathbb{L}$. By a theorem of Cantor that states that any two nonempty countable dense linear orders without end-points are isomorphic[3], there is an order-isomorphism $b : (D, \leq_{\mathbb{L}}) \to (\mathbb{Q}, \leq_{\mathbb{Q}})$.

Claim. *If $x \in \mathbb{L}$, then $D_x = \{b(d) \mid d \in D, d <_{\mathbb{L}} x\}$ is a Dedekind cut in $\mathbb{Q}$.* □

Given $x \in \mathbb{L}$, we let $r(x)$ denote the unique real number corresponding to D_x. The resulting function $r : \mathbb{L} \to \mathbb{R}$ is injective, order-preserving and it extends b.

Claim. *r is surjective.*

Proof. Assume that there is $z \in \mathbb{R} \setminus \mathrm{range}(r)$.
Set

$$U = \{x \in \mathbb{L} \mid r(x) < z\} = \{x \in \mathbb{L} \mid \exists d \in D(x \leq_{\mathbb{L}} d \wedge b(d) < z)\}$$

and $V = \{x \in \mathbb{L} \mid z < r(x)\}$. Then both sets are open with respect to the order topology of $\mathbb{L}$ and $\mathbb{L}$ is the disjoint union of U and V. There are $c, d \in D$ with $b(c) < z < b(d)$. Then $c \in U \neq \emptyset$ and $d \in V \neq \emptyset$, contradiction. □

Definition 5.4.2. A topological space X satisfies the *countable chain condition* ("X is c.c.c") if every set that consists of pairwise disjoint nonempty open sets is countable.

Proposition 5.4.3. *Every separable topological space X satisfies the countable chain condition.*

Proof. Let D be a countable dense subset ob X and let $\mathcal{A}$ be a set of pairwise disjoint nonempty open subsets of X.
Set $\tilde{D} = \{d \in D \mid \exists U \in \mathcal{A}(d \in U)\}$. Given $d \in \tilde{D}$, there is a unique $U_d \in \mathcal{A}$ with $d \in U_d$.
This shows that there is a surjection $s \colon \tilde{D} \to \mathcal{A}, d \mapsto U_d$.
Hence $\mathcal{A}$ is countable. □

Definition 5.4.4. The *Suslin Hypothesis* (SH) is the statement that, up to isomorphism, the real line is the unique dense linear order without end-points where its order topology is connected and satisfies the countable chain condition.

[3]A proof can be found in any textbook on mathematical logic or model theory. This theorem was also proven in the lecture on mathematical logic that was held in summer 2014 at Bonn University.

Remark 5.4.5. Next semester, in the Models of Set Theory I course, we will prove that if ZFC is consistent, then SH is independent from ZFC.

Definition 5.4.6. A *Suslin line* is a linear order $\mathbb{L}$ with the following properties:

1. $\mathbb{L}$ is dense and has no end-points.
2. The order topology of $\mathbb{L}$ is connected.
3. $\mathbb{L}$ satisfies the countable chain condition.
4. The order topology is not separable, i.e. there is no countable dense subset of $\mathbb{L}$.

SH now reformulates to the statement that there is no Suslin line.

We will show that SH is equivalent to the non-existence of certain Aronszajn trees.

Definition 5.4.7. Let $\mathbb{T}$ be a tree.

1. We say that $A \subseteq \mathbb{T}$ is an *antichain in* $\mathbb{T}$ if A consists of pairwise incompatible elements.
2. Given an infinite cardinal κ, we say that $\mathbb{T}$ is a κ*-Suslin tree* if all chains and antichains in $\mathbb{T}$ have cardinality less than κ. Again, if κ is omitted, we mean ω_1-Suslin trees.

Proposition 5.4.8. *If κ is an infinite cardinal and $\mathbb{T}$ is a κ-Suslin tree, then $\mathbb{T}$ is a κ-Aronszajn tree.*

The following fundamental result motivates the above definition.

Theorem 5.4.9 (Kurepa)**.** *The following statements are equivalent:*

1. SH.
2. *There is no Suslin tree.*

Proof of the implication (2) ⇒ (1) of Theorem 5.4.9. Let $\mathbb{L}$ be a Suslin tree. Then, for every countable subset $D \subseteq \mathbb{L}$, there are $x, y \in \mathbb{L}$ with $x <_{\mathbb{L}} y$ and $D \cap [x, y] = \emptyset$. Lecture 21 22nd Dec
This allows us to inductively construct sequences $(a_\alpha)_{\alpha<\omega_1}, (b_\alpha)_{\alpha<\omega_1}, a_\alpha, b_\alpha \in \mathbb{L}$ for all $\alpha < \omega_1$, such that the following statements hold for all $\alpha < \omega_1$:

(a) $a_\alpha <_{\mathbb{L}} b_\alpha$.

(b) $\forall \overline{\alpha} < \alpha\ a_{\overline{\alpha}}, b_{\overline{\alpha}} \notin [a_\alpha, b_\alpha]$.

Claim. *If $\overline{\alpha} < \alpha < \omega_1$, then either $[a_\alpha, b_\alpha] \subseteq (a_{\overline{\alpha}}, b_{\overline{\alpha}})$ or $[a_\alpha, b_\alpha] \cap [a_{\overline{\alpha}}, b_{\overline{\alpha}}] = \emptyset$.*

Proof. Since $a_{\overline{\alpha}}, b_{\overline{\alpha}} \notin [a_\alpha, b_\alpha]$, we either have

- $a_{\overline{\alpha}} <_{\mathbb{L}} a_\alpha <_{\mathbb{L}} b_\alpha <_{\mathbb{L}} b_{\overline{\alpha}}$ or

- $a_\alpha <_{\mathbb{L}} b_\alpha <_{\mathbb{L}} a_{\overline{\alpha}} <_{\mathbb{L}} b_{\overline{\alpha}}$ or
- $a_{\overline{\alpha}} <_{\mathbb{L}} b_{\overline{\alpha}} <_{\mathbb{L}} a_\alpha <_{\mathbb{L}} b_\alpha$.

□

Define a binary relation $\leq_{\mathbb{T}}$ on ω_1 by setting

$$\overline{\alpha} \leq_{\mathbb{T}} \alpha \text{ if and only if } \overline{\alpha} = 0 \vee \alpha = \overline{\alpha} \vee \big(\overline{\alpha} < \alpha \wedge [a_\alpha, b_\alpha] \subseteq (a_{\overline{\alpha}}, b_{\overline{\alpha}})\big)$$

for all $\alpha, \overline{\alpha} < \omega_1$ and set $\mathbb{T} = (\omega_1, \leq_{\mathbb{T}})$.
Then $\mathbb{T}$ is a partial order with minimal element 0 and $(\mathrm{pred}_{\mathbb{T}}(\alpha), <_{\mathbb{T}})$ is well-founded for all $\alpha < \omega_1$ (otherwise, there would be infinite descendng chains of ordinals, contradiction).

Claim. *If $\alpha < \omega_1$, then $\mathrm{pred}_{\mathbb{T}}(\alpha)$ is linearly ordered by $\leq_{\mathbb{T}}$.*

Proof. Choose $\alpha_0, \alpha_1 \in \mathrm{pred}_{\mathbb{T}}(\alpha)$ with $\alpha_0 < \alpha_1$. We may also assume $\alpha_i \neq 0$.
Then we have $\emptyset \neq [a_\alpha, b_\alpha] \subseteq (a_{\alpha_0}, b_{\alpha_0}) \cup (a_{\alpha_1}, b_{\alpha_1})$ and the above claim shows that $[a_{\alpha_1}, b_{\alpha_1}] \subseteq (a_{\alpha_0}, b_{\alpha_0})$. This shows that $\alpha_0 <_{\mathbb{T}} \alpha_1 <_{\mathbb{T}} \alpha$. □

This shows that $\mathbb{T}$ is a tree.

Claim. *All antichains in $\mathbb{T}$ are countable.*

Proof. Let $(\alpha_\gamma)_{\gamma<\lambda}$ be an injective enumeration of an antichain in $\mathbb{T}$. By the definition of $\mathbb{T}$, we have $(a_{\alpha_\gamma}, b_{\alpha,\gamma}) \cap (a_{\alpha_{\overline{\gamma}}}, b_{\alpha_{\overline{\gamma}}}) = \emptyset$ for all $\gamma, \overline{\gamma} < \lambda$ with $\gamma \neq \overline{\gamma}$. But now our assumption that $\mathbb{L}$ is a Suslin line implies that λ is countable. □

Claim. *All chains in $\mathbb{T}$ are countable.*

Proof. Let c be a chain in $\mathbb{T}$ with $\mathrm{lh}_{\mathbb{T}}(c) = \lambda$.
Then $c(\overline{\gamma}) < c(\gamma), a_{c(\gamma)} \neq a_{c(\overline{\gamma})}$ and $[a_{c(\gamma), b_{c(\gamma)}}] \subseteq (a_{c(\overline{\gamma})}, b_{c(\overline{\gamma})})$ for all $\overline{\gamma} < \gamma < \lambda$.
In particular, we have $(a_{c(\overline{\gamma})}, a_{c(\overline{\gamma}+1)}) = \emptyset$ for all $\overline{\gamma} < \gamma < \lambda$. By the above claim, we know that λ is countable. □

This shows that $\mathbb{T}$ is a Suslin tree. □

Definition 5.4.10. Let $\mathbb{T}$ be a tree and $\preceq$ be a linear order on $\mathbb{T}$.

1. We say that $\preceq$ is *suitable* if $s \prec t$ holds for all $s, t \in \mathbb{T}$ with $\mathrm{lh}_{\mathbb{T}}(s) < \mathrm{lh}_{\mathbb{T}}(t)$.
2. Assume that $\preceq$ is suitable for $\mathbb{T}$. We define a binary relation $\preceq_{\mathrm{lex}}$ on $\partial\mathbb{T}$ by setting

$$b \preceq_{\mathrm{lex}} c \text{ if and only if } b = c \vee \big(b \neq c \wedge b(\Delta(b,c)) \prec c(\Delta(b,c))\big)$$

for all $b, c \in \partial\mathbb{T}$.

Proposition 5.4.11. *If $\preceq$ is a suitable linear order on a tree $\mathbb{T}$, then $(\partial\mathbb{T}, \preceq_{\mathrm{lex}})$ is a linear order.*

Proof. Choose $b, c \in \partial\mathbb{T}$ with $b \neq c$ and set $\delta = \Delta(b, c)$. Then $b(\delta) \neq c(\delta)$ and therefore either $b(\delta) \prec c(\delta)$ or $c(\delta) \prec b(\delta)$. Hence either $b \preceq_{\text{lex}} c$ or $c \preceq_{\text{lex}} b$. Clearly, if both $b \preceq_{\text{lex}} c$ and $c \preceq_{\text{lex}} b$ hold, then we have $b = c$.
Finally, choose $b, c \in \partial\mathbb{T}$ with $b \preceq_{\text{lex}} c$ and $c \preceq_{\text{lex}} d$. We may assume $b \neq c \neq d \neq b$.

Case 1: $\delta = \Delta(b, c) = \Delta(c, d)$. Then $\delta = \Delta(b, d), b(\delta) \prec c(\delta) \prec d(\delta)$ and $b \preceq_{\text{lex}} d$.

Case 2: $\delta = \Delta(b, c) < \Delta(c, d)$. Then $\delta = \Delta(b, d), b(\delta) \prec c(\delta) = d(\delta)$ and $b \preceq_{\text{lex}} d$.

Case 3: $\Delta(b, c) > \Delta(c, d) = \delta$. Then $\delta = \Delta(b, d), b(\delta) = c(\delta) \prec d(\delta)$ and $b \preceq_{\text{lex}} d$.

□

Proposition 5.4.12. *Let $\mathbb{T}$ be a tree that is extensional at limit levels and $\preceq$ bea suitable linear ordering of $\mathbb{T}$ with the property that $(\mathrm{succ}_{\mathbb{T}}(t), \preceq)$ is a dense linear order without end-points for every $t \in \mathbb{T}$.*
Then $(\partial\mathbb{T}, \preceq_{\text{lex}})$ is a dense linear order without end-points.

Proof. Choose $b, c \in \partial\mathbb{T}$ with $b \prec_{lex} c$ and set $\delta = \Delta(b, c)$. Since $\mathbb{T}$ is extensional at limit levels, we have $\delta = \bar{\delta} + 1$ and there are $t_0, t_1, t_2 \in \mathrm{succ}_{\mathbb{T}}(b(\bar{\delta}))$ with $t_0 \prec b(\delta) \prec t_1 \prec c(\delta) \prec t_2$ by our assumption.
By the Hausdorff Maximality Principle (Lemma 5.1.9 (2)), there are $b_0, b_1, b_2 \in \partial\mathbb{T}$ with $t_i \in b_i, b_0 \prec_{lex} b \prec_{lex} b_1 \prec_{lex} b_2$. Since our assumptions imply that $\partial\mathbb{T}$ is infinite, this shows the proposition. □

Proposition 5.4.13. *Let $\mathbb{L}$ be a dense linear ordering without end-points and let $\mathbb{K}$ denote its completion (i.e. the set of Dedekind cuts in $\mathbb{L}$ ordered by inclusion).*

1. *$\mathbb{K}$ is a dense linear order without end-points when the order topology is connected.*
2. *If the order topology of $\mathbb{L}$ satisfies the countable chain condition, then the same is true for $\mathbb{K}$.*
3. *If the order topology of $\mathbb{L}$ is not separable, then the same is true for $\mathbb{K}$.*

Proof.
1. This follows from the definition of $\mathbb{K}$.
2. Let $(U_\alpha)_{\alpha<\lambda}$ be an injective enumeration of pairwise disjoint nonempty open subsets of $\mathbb{K}$. Given $\alpha < \lambda$, there are $a_\alpha, b_\alpha \in \mathbb{L}$ with $a_\alpha <_{\mathbb{L}} b_\alpha$ and $(a_\alpha, b_\alpha)_{\mathbb{K}} \subseteq U_\alpha$.
Then $\big((a_\alpha, b_\alpha)_{\mathbb{L}}\big)_{\alpha<\lambda}$ is an injective enumeration of pairwise disjoint nonempty open subsets of $\mathbb{L}$ and λ is countable.
3. Assume that D is a countable dense subset of $\mathbb{K}$. Then there is a countable subset E of $\mathbb{L}$ such that for all $x, y \in \mathbb{K}$ with $x <_{\mathbb{K}} y$ there is $e \in E$ with $x <_{\mathbb{K}} e <_{\mathbb{K}} y$. Then E is dense in $\mathbb{L}$, contradiction.

□

Lemma 5.4.14. *Let κ be an uncountable regular cardinal and $\mathbb{T}'$ be a κ-Aronszajn tree. Then there is a κ-Aronszajn tree $\mathbb{T}$ with the following properties:*

1. *$\mathbb{T}$ is extensional at limit levels.*

2. *If $s \in \mathbb{T}$ and $\mathrm{lh}_{\mathbb{T}}(s) \leq \alpha < \kappa$, then $\mathbb{T}(\alpha) \cap \mathbb{T}_s \neq \emptyset$.*

3. *If $t \in \mathbb{T}$, then $\mathrm{succ}_{\mathbb{T}}(t)$ is an infinite set.*

4. *If $\mathbb{T}'$ is a κ-Suslin tree, then so is $\mathbb{T}$.*

Proof. Define
$$\mathbb{T} = \{t \in \mathbb{T}' \mid \forall\, \mathrm{lh}_{\mathbb{T}'}(t) \leq \alpha < \kappa\; \mathbb{T}'(\alpha) \cap \mathbb{T}'_t \neq \emptyset\}.$$
Then $\mathrm{root}(\mathbb{T}') \in \mathbb{T}$ and $\mathbb{T}$ is $\leq_{\mathbb{T}'}$-downwards closed in $\mathbb{T}'$.
Hence $\mathbb{T} = (\mathbb{T}, \leq_{\mathbb{T}'})$ is a tree.

Claim. $\mathrm{ht}(\mathbb{T}_t) = \kappa$ *for all* $t \in \mathbb{T}$.

Proof. Assume that $\mathrm{ht}(\mathbb{T}_t) = \alpha < \kappa$. Then for $s \in \mathbb{T}'(\alpha) \cap \mathbb{T}'_t$, there is $\alpha < \beta_s < \kappa$ with $\mathbb{T}'(\beta_s) \cap \mathbb{T}'_s = \emptyset$.
By assumption, there is $\beta < \kappa$ with $\beta_s < \beta$ for all $s \in \mathbb{T}'(\alpha) \cap \mathbb{T}'_t$.
Choose $s \in \mathbb{T}'(\beta) \cap \mathbb{T}'_t$. Then $s \restriction \alpha \in \mathbb{T}'(\alpha) \cap \mathbb{T}'_t$ and $s \restriction \beta_{s\restriction\alpha} \in \mathbb{T}'(\beta_{s\restriction\alpha}) \cap \mathbb{T}'_{s\restriction\alpha} = \emptyset$, contradiction. $\square$

Since $\mathbb{T}'$ is a κ-Aronszajn tree, this claim shows that $\mathbb{T}$ is a κ-Aronszajn tree.

Claim. *If $t \in \mathbb{T}$, then there is $\mathrm{lh}_{\mathbb{T}}(t) \leq \alpha < \kappa$ with $|\mathbb{T}(\alpha) \cap \mathbb{T}_t| > 1$.*

Proof. By the above claim, we have $\mathbb{T}(\alpha) \cap \mathbb{T}_t \neq \emptyset$ for all $\mathrm{lh}_{\mathbb{T}}(t) < \alpha < \kappa$. But if $|\mathbb{T}(\alpha) \cap \mathbb{T}_t| = 1$ for all $\mathrm{lh}_{\mathbb{T}}(t) \leq \alpha < \kappa$, then there would be some $b \in [\mathbb{T}] = \emptyset$, contradiction. $\square$

Claim. *If $t \in \mathbb{T}$, then there is $\mathrm{lh}_{\mathbb{T}}(t) < \alpha < \kappa$ such that $|\mathbb{T}(\beta) \cap \mathbb{T}_t| \geq \omega$ for all $\alpha \leq \beta < \kappa$.*

Proof. Directly by the previous two claims. $\square$

By the above claims, there is a club set C in κ such that $|\mathbb{T}(\alpha) \cap \mathbb{T}_t| \geq \omega$ for all $t \in \mathbb{T}$ with $\mathrm{lh}_{\mathbb{T}}(t) \in C$ and $\mathrm{lh}_{\mathbb{T}}(t) < \alpha \in C$.
Set
$$\mathbb{T}_* = \left(\bigcup_{\alpha \in C} \mathbb{T}(\alpha), \leq_{\mathbb{T}} \right).$$
Then $\mathbb{T}_*$ is a κ-Aronszajn tree with properties (2) and (3).
Since every antichain in $\mathbb{T}_*$ is also an antichain in $\mathbb{T}'$, $\mathbb{T}_*$ also satisfies property (4).
Finally, it is easy to see that the tree $\overline{\mathbb{T}_*}$ constructed in Lemma 5.1.14 satisfies all four properties. $\square$

Proof of the implication (1) ⇒ (2) of Theorem 5.4.9. Let $\mathbb{T}_0$ be a Suslin tree. By the previous lemma, there is a Suslin tree $\mathbb{T}$ with the discussed properties and we can find a suitable linear order $\preceq$ such that $(\mathrm{succ}_{\mathbb{T}}(t), \preceq)$ is a dense linear order without end-points for every $t \in \mathbb{T}$. Then $\mathbb{L} = (\partial\mathbb{T}, \preceq_{\mathrm{lex}})$ is a dense linear order without end-points.

Claim. *The order topology of $\mathbb{L}$ satisfies the countable chain condition.*

Proof. Let $(U_\alpha)_{\alpha<\lambda}$ be an injective enumeration of pairwise disjoint nonempty open subsets of $\mathbb{L}$. Given $\alpha < \lambda$, there are $a_\alpha, b_\alpha \in \partial\mathbb{T}$ with $a_\alpha \prec_{lex} b_\alpha, (a_\alpha, b_\alpha) \subseteq U_\alpha$ and $\Delta(a_\alpha, b_\alpha) = \delta_\alpha + 1$ for $\delta_\alpha < \omega_1$. Choose $t_\alpha \in \mathbb{T}(\delta_\alpha+1)$ with $a_\alpha(\delta_\alpha+1) \prec t_\alpha \prec b_\alpha(\delta_\alpha+1)$. Then $(t_\alpha)_{\alpha<\omega_1}$ is an enumeration of an antichain in $\mathbb{T}$ and λ is countable. □

Claim. *The order topology of $\mathbb{L}$ is not separable.*

Proof. Let $D \subseteq \mathbb{L}$ be countable. Since $[\mathbb{T}] = \emptyset$, there is some $\alpha < \omega_1$ with $\mathrm{lh}_\mathbb{T}(b) < \alpha$ for all $b \in D$. Choose $t \in \mathbb{T}(\alpha)$, $t_0, t_1 \in \mathrm{succ}_\mathbb{T}(t)$, $b_0, b_1 \in \partial\mathbb{T}$ with $t_0 \prec t_1$ and $t_i \in b_i$.
If D is dense, then there is $b \in D \cap (b_0, b_1)$ and $t \in b$.
Then $\alpha = \mathrm{lh}_\mathbb{T}(t) < \mathrm{lh}_\mathbb{T}(b)$, contradiction. □

By Proposition 5.4.13, the completion of $\mathbb{L}$ is a Suslin line. □

5.5 Aronszajn Trees and Walks on Ordinals

By König's Lemma (Theorem 5.2.1), there are no ω-Aronszajn trees. In contrast to this, the following fundamental theorems show that there are κ-Aronszajn trees for $\kappa = \omega_1$ and certain other uncountable cardinals. Lecture 22 7th Jan

Theorem 5.5.1 (Aronszajn)**.** *There is a ω_1-Aronszajn tree.*

Theorem 5.5.2 (Specker)**.** *If κ is an infinite cardinal with $\kappa = \kappa^{<\kappa}$, then there is a κ^+-Aronszajn tree.*

The theorem of Specker implies the theorem of Aronszajn, since ω has the property $\omega = \omega^{<\omega}$. There are models of ZFC in which the statement holds that ω is the only cardinal with this property. We will present modern proofs of these results using the method of "walks on ordinals" invented by TODORCEVIC [12], one of KUREPA's students.

Definition 5.5.3 (Todorcevic)**.** Let κ e an uncountable regular cardinal.
We say that a sequence $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ is a *C-sequence* if the following statements hold for all $\alpha < \kappa$:

1. $C_{\alpha+1} = \{\alpha\}$.
2. If α is a limit ordinal, then C_α is a closed unbounded subset of α.

The existence of C-sequences is given by the Axiom of Choice.

Proposition 5.5.4. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence and $\alpha < \beta < \kappa$.*
Then there is a unique finite sequence $(\gamma_0, \ldots, \gamma_n)$ of ordinals less than κ with the following properties:

1. *$\gamma_0 = \beta$ and $\gamma_n = \alpha$.*
2. *If $i < n$, then $\gamma_{i+1} = \min(C_{\gamma_i} \setminus \alpha)$.*

Proof. By the above definition, there can be at most one sequence with these properties. We inductively construct a sequence $(\gamma_i)_{i<\omega}, \gamma_i < \kappa$ for all $i < \omega$, such that $\gamma_0 = \beta$ and the following statements hold for all $i < \omega$:

1. If $\gamma_i > \alpha$, then $\gamma_{i+1} = \min(C_{\gamma_i} \setminus \alpha) \geq \alpha$.

2. If $\gamma_i \leq \alpha$, then $\gamma_i = \gamma_{i+1}$.

By the Foundation Axiom, there is a minimal $n < \omega$ with $\gamma_n = \gamma_{n+1} = \alpha$.
Then the sequence $(\gamma_0, \ldots, \gamma_n)$ satisfies the above properties. □

Definition 5.5.5 (Todorcevic)**.** Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence.

1. Given $\alpha < \beta < \kappa$, we call the unique finite sequence given by the previous proposition the *walk from β to α through $\vec{C}$*.

2. Let $\alpha < \beta < \kappa$ and $(\gamma_0, \ldots, \gamma_n)$ be the walk from β to α through $\vec{C}$. Define

$$\rho_{\vec{C}}(\alpha, \beta) = \big(\operatorname{type}(C_{\gamma_0} \cap \alpha), \ldots, \operatorname{type}(C_{\gamma_{n-1}} \cap \alpha)\big) \in {}^{<\omega}\kappa.$$

   This finite sequence is called the *full code of the walk from β to α through $\vec{C}$.*

3. Given $\beta < \kappa$, define
$$\rho_{\vec{C}}(\cdot, \beta) \colon \beta \to {}^{<\omega}\kappa, \alpha \mapsto \rho_{\vec{C}}(\alpha, \beta).$$

4. Define
$$\mathbb{T}(\rho_{\vec{C}}) = \big(\{\rho_{\vec{C}}(\cdot, \beta)\restriction\alpha \mid \alpha \leq \beta < \kappa\}, \subseteq\big).$$

Proposition 5.5.6. *If $\vec{C}$ is a C-sequence of length κ, then $\mathbb{T}(\rho_{\vec{C}})$ is a tree of height κ with levels*
$$\mathbb{T}(\rho_{\vec{C}})(\alpha) = \{\rho_{\vec{C}}(\cdot, \beta)\restriction\alpha \mid \alpha \leq \beta < \kappa\}$$
for all $\alpha < \kappa$.

Proposition 5.5.7. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence. Given $\alpha < \beta < \kappa$, the following statements are equivalent:*

1. $\alpha \in C_\beta$.

2. The sequence $\rho_{\vec{C}}(\alpha, \beta)$ has length 1.

Proof.

(1) ⇒ (2): If $\alpha \in C_\beta$, then $\alpha = \min(C_\beta \setminus \alpha)$, the walk from β to α through $\vec{C}$ is (β, α) and $\rho_{\vec{C}}(\alpha, \beta) = \big(\operatorname{type}(C_\beta \cap \alpha)\big)$.

(2) ⇒ (1): If $\rho_{\vec{C}}(\alpha, \beta)$ has length 1, then the walk from β to α through $\vec{C}$ has length 2 and $\alpha = \min(C_\beta \setminus \alpha) \in C_\beta$.

□

Lemma 5.5.8. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence, $\overline{\alpha} \leq \alpha \leq \beta < \kappa$ and let $(\gamma_0, \ldots, \gamma_n)$ denote the walk from β to α through $\vec{C}$.*
Then the following statements are equivalent:

1. *α appears in the walk from β to $\overline{\alpha}$ through $\vec{C}$.*
2. *The sequence $(\gamma_0, \ldots, \gamma_n)$ is an initial segment of the walk from β to $\overline{\alpha}$ through $\vec{C}$.*
3. *If $i < n$, then $C_{\gamma_i} \cap [\overline{\alpha}, \alpha) = \emptyset$.*

Proof.

(2) $\Leftrightarrow$ (1): γ_n is α.

(3) $\Rightarrow$ (2): By induction on $i \leq n$, we show that γ_i is the i-th ordinal in the walk from β to $\overline{\alpha}$ through $\vec{C}$.

$i = 0$: $\checkmark$.

$i \to i+1$: Assume this statement holds for $i < n$.
By assumption, $\gamma_{i+1} = \min(C_{\gamma_i} \setminus \alpha) = \min(C_{\gamma_i} \setminus \overline{\alpha})$ and this shows that γ_{i+1} is the $(i+1)$-th step in the walk from β to $\overline{\alpha}$ through $\vec{C}$.

This shows that (2) holds.

(1) $\Rightarrow$ (3): Assume (1) holds and (3) fails. Let $i < n$ be the minimal counterexample to (3), i.e. $i < n$ is minimal with $C_{\gamma_i} \cap [\overline{\alpha}, \alpha) \neq \emptyset$.
By the above computations, we know that $(\gamma_0, \ldots, \gamma_i)$ is an initial segment of the walk from β to $\overline{\alpha}$ through $\vec{C}$. Then $\gamma_i > \overline{\alpha}$ and our assumptions imply that $\min(C_{\gamma_i} \setminus \overline{\alpha}) < \alpha$. This shows that α does not appear in the walk from β to $\overline{\alpha}$ through $\vec{C}$, contradicting (1).

□

Lemma 5.5.9. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence and $\overline{\alpha} \leq \alpha \leq \beta < \kappa$. Then the following statements are equivalent:*

1. *α appears in the walk from β to $\overline{\alpha}$ through $\vec{C}$.*
2. *If $\overline{\alpha} < \alpha < \beta$, then $\rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\alpha, \beta)^\frown \rho_{\vec{C}}(\overline{\alpha}, \alpha)$, i.e. the latter sequence directly after the former.*
3. *If $\alpha < \beta$, then $\rho_{\vec{C}}(\alpha, \beta)$ is an initial segment of $\rho_{\vec{C}}(\overline{\alpha}, \beta)$.*

Proof. Let $(\gamma_0, \ldots, \gamma_n)$ denote the walk from β to α through $\vec{C}$ and $(\overline{\gamma}_0, \ldots, \overline{\gamma}_m)$ denote the walk from β to $\overline{\alpha}$ through $\vec{C}$.

(3) $\Leftrightarrow$ (2): $\checkmark$.

(2) $\Rightarrow$ (1): Assume (2) holds and (1) fails. Then $\alpha < \beta, n \leq m$ and there is a minimal $i < n$ with $\gamma_{i+1} \neq \overline{\gamma}_{i+1}$. Then $\gamma_i = \overline{\gamma}_i, \min(C_{\gamma_i} \setminus \alpha) \neq \min(C_{\gamma_i} \setminus \overline{\alpha})$ and $C_{\gamma_i} \cap \overline{\alpha} \subsetneq C_{\gamma_i} \cap \alpha$. In particular, we have $\operatorname{type}(C_{\gamma_i} \cap \alpha) \neq \operatorname{type}(C_{\gamma_i} \cap \overline{\alpha})$ and hence $\rho_{\vec{C}}(\alpha, \beta)(i) \neq \rho_{\vec{C}}(\overline{\alpha}, \beta)(i)$, contradiction.

(1) $\Rightarrow$ (2): Assume that (1) holds and (2) fails. Then $\overline{\alpha} < \alpha < \beta$ and there is $k \leq m$ with $\alpha = \overline{\gamma}_k$. By the previous lemma, we have $k = n, \gamma_i = \overline{\gamma}_i$ for all $i < n$ and $C_{\gamma_i} \cap [\overline{\alpha}, \alpha) = \emptyset$ for all $i < n$. Then

$$\rho_{\vec{C}}(\alpha, \beta)(i) = \operatorname{type}(C_{\gamma_i} \cap \alpha) = \operatorname{type}(C_{\gamma_i} \cap \overline{\alpha}) = \rho_{\vec{C}}(\overline{\alpha}, \beta)(i),$$

for all $i < n$.
Since $(\overline{\gamma}_n, \ldots, \overline{\gamma}_m)$ is the walk from α to $\overline{\alpha}$ through $\vec{C}$, we have

$$\rho_{\vec{C}}(\overline{\alpha}, \alpha)(i) = \operatorname{type}(C_{\overline{\gamma}_{n+i}} \cap \overline{\alpha}) = \rho_{\vec{C}}(\overline{\alpha}, \beta)(n+i),$$

for all $i < m - n$, contradiction.

□

Lemma 5.5.10. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence, $\alpha \in \operatorname{Lim} \cap \kappa$ and $\alpha \leq \overline{\beta} \leq \beta < \kappa$. We let $(\gamma_0, \ldots, \gamma_n)$ denote the walk from β to α through $\vec{C}$ and let $(\overline{\gamma}_0, \ldots, \overline{\gamma}_m)$ denote the walk from $\overline{\beta}$ to α through $\vec{C}$. If $m = n$ and $C_{\gamma_i} \cap \alpha = C_{\overline{\gamma}_i} \cap \alpha$ holds for al $i \leq n$, then $\rho_{\vec{C}}(\cdot, \beta) \restriction \alpha = \rho_{\vec{C}}(\cdot, \overline{\beta}) \restriction \alpha$.* Lecture 23 12th Jan

Proof. Choose $\overline{\alpha} < \alpha$ and let $k \leq n$ be minimal with $C_{\gamma_k} \cap [\overline{\alpha}, \alpha) \neq \emptyset$. By Lemma 5.5.8, we know that $(\gamma_0, \ldots, \gamma_k)$ is an initial segment of the walk from β to $\overline{\alpha}$ through $\vec{C}$ and the next step in this walk is $\gamma_* = \min(C_{\gamma_k} \setminus \overline{\alpha})$.
By the assumption, we know that $k = \min\{i \leq n \mid C_{\overline{\gamma}_i} \cap [\overline{\alpha}, \alpha) \neq \emptyset\}$, $(\overline{\gamma}_0, \ldots, \overline{\gamma}_n)$ is an initial segment of the walk from $\overline{\beta}$ to $\overline{\alpha}$ through $\vec{C}$ and $\gamma_* = \min(C_{\overline{\gamma}_k} \setminus \overline{\alpha})$ is the next step in the walk.
If $\gamma_* = \overline{\alpha}$, then

$$\begin{aligned}
\rho_{\vec{C}}(\overline{\alpha}, \beta) &= \big(\operatorname{type}(C_{\gamma_0} \cap \overline{\alpha}), \ldots, \operatorname{type}(C_{\gamma_k} \cap \overline{\alpha})\big) \\
&= \big(\operatorname{type}(C_{\overline{\gamma}_0} \cap \overline{\alpha}), \ldots, \operatorname{type}(C_{\overline{\gamma}_k} \cap \overline{\alpha})\big) \\
&= \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta}).
\end{aligned}$$

If $\gamma_* > \overline{\alpha}$, then the previous lemma implies that

$$\begin{aligned}
\rho_{\vec{C}}(\overline{\alpha}, \beta) &= \rho_{\vec{C}}(\gamma_*, \beta) ^\frown \rho_{\vec{C}}(\overline{\alpha}, \gamma_*) \\
&= \big(\operatorname{type}(C_{\gamma_0} \cap \gamma_*), \ldots, \operatorname{type}(C_{\gamma_k} \cap \gamma_*)\big) ^\frown \rho_{\vec{C}}(\overline{\alpha}, \gamma_*) \\
&= \big(\operatorname{type}(C_{\overline{\gamma}_0} \cap \gamma_*), \ldots, \operatorname{type}(C_{\overline{\gamma}_k} \cap \gamma_*)\big) ^\frown \rho_{\vec{C}}(\overline{\alpha}, \gamma_*) \\
&= \rho_{\vec{C}}(\gamma_*, \overline{\beta}) ^\frown \rho_{\vec{C}}(\overline{\alpha}, \gamma_*) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta}).
\end{aligned}$$

Hence $\rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta})$. □

Lemma 5.5.11. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence with the property that the set $\{C_\beta \cap \alpha \mid \alpha \leq \beta < \kappa\}$ has cardinality less than κ for every $\alpha \in \mathrm{Lim} \cap \kappa$.*
Then $|\mathbb{T}(\rho_{\vec{C}})(\alpha)| < \kappa$ for all $\alpha < \kappa$.

Note that the condition is always satisfied if κ is inaccessible.

Proof. We first prove the statement for limit ordinals. Assume there is $\alpha \in \mathrm{Lim} \cap \kappa$ and a sequence $(\beta_\delta)_{\delta<\kappa}$ of ordinals in $[\alpha, \kappa)$ such that

$$\rho_{\vec{C}}(\cdot, \beta_\delta)\restriction\alpha \neq \rho_{\vec{C}}(\cdot, \beta_{\overline{\delta}})\restriction\alpha \text{ for all } \overline{\delta} < \delta < \kappa.$$

Given $\delta < \kappa$, let $(\gamma_0^{(\delta)}, \ldots, \gamma_{n_\delta}^{(\delta)})$ denote the walk from β_δ to α through $\vec{C}$.
By the assumption, there are $\overline{\delta} < \delta < \kappa$ such that $n_\delta = n_{\overline{\delta}}$ and $C_{\gamma_i^{(\delta)}} \cap \alpha = C_{\gamma_i^{(\overline{\delta})}} \cap \alpha$ for all $i \leq n_\delta$.
By the previous lemma, this implies that $\rho_{\vec{C}}(\cdot, \beta_\delta)\restriction\alpha = \rho_{\vec{C}}(\cdot, \beta_{\overline{\delta}})\restriction\alpha$, contradiction.
Now, let $\alpha < \kappa$ and $(\beta_\delta)_{\delta<\kappa}$ be an injective enumeration of ordinals in $[\alpha, \kappa)$. By the above computations, there are $\overline{\delta} < \delta < \kappa$ with $\beta_\delta, \beta_{\overline{\delta}} > \alpha + \omega$ and
$\rho_{\vec{C}}(\cdot, \beta_\delta)\restriction(\alpha + \omega) = \rho_{\vec{C}}(\cdot, \beta_{\overline{\delta}}\restriction(\alpha + \omega)$.
Then $\rho_{\vec{C}}(\cdot, \beta_\delta)\restriction\alpha = \rho_{\vec{C}}(\cdot, \beta_{\overline{\delta}})\restriction\alpha$. □

We will now consider criteria for the non-existence of cofinal branches through $\mathbb{T}(\rho_{\vec{C}})$.

Lemma 5.5.12. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence with $[\mathbb{T}(\rho_{\vec{C}})] \neq \emptyset$. Then there is a club C in κ and $\xi < \kappa$ such that for every $\xi \leq \alpha < \kappa$ there is $\alpha \leq \beta < \kappa$ with $C \cap \alpha = C_\beta \cap [\xi, \alpha)$.*

Proof. Fix $b \in [\mathbb{T}(\rho_{\vec{C}})]$. By Proposition 5.5.6, there is a sequence $(\beta_\alpha)_{\alpha<\kappa}$ such that $b(\alpha) = \rho_{\vec{C}}(\cdot, \beta_\alpha)\restriction\alpha$ for every $\alpha < \kappa$.
Given $\alpha < \kappa$, let $(\gamma_0^{(\alpha)}, \ldots, \gamma_{n_\alpha}^{(\alpha)})$ denote the walk from β_α to α through $\vec{C}$ and let $k_\alpha \leq n_\alpha$ be minimal with $C_{\gamma_{k_\alpha}^{(\alpha)}} \cap \alpha$ is unbounded in α. By Fodor's Lemma (Theorem 4.1.9), there is $\xi < \kappa$, $S \subseteq \kappa \setminus \xi$ unbounded in κ and $k, n < \omega$ such that $n_\alpha = n, k_\alpha = k$ and $C_{\gamma_i^{(\alpha)}} \cap \alpha \subseteq \xi$ for all $i < k$ and $\alpha \in S$.

Claim. *If $\alpha, \overline{\alpha} \in S$ with $\overline{\alpha} < \alpha$, then*

$$C_{\gamma_k^{(\alpha)}} \cap [\xi, \overline{\alpha}) = C_{\gamma_k^{(\overline{\alpha})}} \cap [\xi, \overline{\alpha}).$$

Proof. Choose $\eta \in [\xi, \overline{\alpha})$. Then

$$C_{\gamma_i^{(\alpha)}} \cap [\eta, \gamma_k^{(\alpha)}) = \emptyset = C_{\gamma_k^{(\overline{\alpha})}} \cap [\eta, \gamma_k^{(\alpha)}) \text{ for all } i < k.$$

Hence $\gamma_k^{(\alpha)}$ appears in the walk from β_α to η through $\vec{C}$ and $\gamma_k^{(\overline{\alpha})}$ appears in the walk from $\beta_{\overline{\alpha}}$ to η through $\vec{C}$.

If $k = 0$, then $\rho_{\vec{C}}(\eta, \gamma_k^{(\alpha)}) = \rho_{\vec{C}}(\eta, \beta_\alpha) = \rho_{\vec{C}}(\eta, \beta_{\overline{\alpha}}) = \rho_{\vec{C}}(\eta, \gamma_k^{(\overline{\alpha})})$.
If $k > 0$, then we can use Lemma 5.5.9 to conclude that

$$\rho_{\vec{C}}(\gamma_k^{(\alpha)}, \beta_\alpha) ^\frown \rho_{\vec{C}}(\eta, \gamma_k^{(\alpha)}) = \rho_{\vec{C}}(\eta, \beta_\alpha) = \rho_{\vec{C}}(\eta, \beta_{\overline{\alpha}}) = \rho_{\vec{C}}(\gamma_k^{(\overline{\alpha})}, \beta_{\overline{\alpha}}) ^\frown \rho_{\vec{C}}(\eta, \gamma_k^{(\overline{\alpha})})$$

and this implies that $\rho_{\vec{C}}(\eta, \gamma_k^{(\alpha)}) = \rho_{\vec{C}}(\eta, \gamma_k^{(\overline{\alpha})})$, because the sequences $\rho_{\vec{C}}(\gamma_k^{(\alpha)}, \beta_\alpha)$ and $\rho_{\vec{C}}(\gamma_k^{(\overline{\alpha})}, \beta_{\overline{\alpha}})$ both have length $k - 1$.
This shows that $\rho_{\vec{C}}(\eta, \gamma_k^{(\alpha)}) = \rho_{\vec{C}}(\eta, \gamma_k^{(\overline{\alpha})})$ holds, and by Proposition 5.5.7 we have

$$\begin{aligned} \eta \in C_{\gamma_k^{(\alpha)}} \quad & \text{if and only if} \quad \rho_{\vec{C}}(\eta, \gamma_k^{(\alpha)}) \text{ has length } 1 \\ & \text{if and only if} \quad \rho_{\vec{C}}(\eta, \gamma_k^{(\overline{\alpha})}) \text{ has length } 1 \\ & \text{if and only if} \quad \eta \in C_{\gamma_k^{(\overline{\alpha})}}. \end{aligned}$$

This shows that $C_{\gamma_k^{\alpha)}} \cap [\xi, \alpha) = C_{\gamma_k^{(\overline{\alpha})}} \cap [\xi, \overline{\alpha})$. □

Define

$$C = \bigcup \{ C_{\gamma_k^{(\alpha)}} \cap [\xi, \alpha) \mid \alpha \in S \}.$$

Claim. *C is a club in κ with $C \cap \alpha = C_{\gamma_k^{(\alpha)}} \cap [\xi, \alpha)$.*

Proof. Choose $\eta < \kappa$. Then there is $\alpha \in S$ with $\eta < \alpha$. Since $C_{\gamma_k^{(\alpha)}}$ is unbounded in α, we have $C_{\gamma_k^{(\alpha)}} \cap [\eta, \alpha) \neq \emptyset$ and hence $C \setminus \eta \neq \emptyset$. This shows that C is unbounded in κ.
Choose $\alpha \in S$. Then the above claim shows that

$$\begin{aligned} C \cap \alpha &= \bigcup \{ C_{\gamma_k^{(\overline{\alpha})}} \cap [\xi, \overline{\alpha}) \mid \overline{\alpha} \in S \cap \alpha \} \cup \bigcup \{ C_{\gamma_k^{(\beta)}} \cap [\xi, \alpha) \mid \beta \in S \setminus \alpha \} \\ &= C_{\gamma_k^{(\alpha)}} \cap [\xi, \alpha). \end{aligned}$$

Finally, let $\overline{\alpha}$ be a limit point of C and $\alpha \in S$ with $\overline{\alpha} < \alpha$. Since $C \cap \alpha = C_{\gamma_k^{(\alpha)}} \cap [\xi, \alpha)$ and $\overline{\alpha} > \xi$, we know that $\overline{\alpha}$ is a limit point of $C_{\gamma_k^{(\alpha)}}$ and hence $\overline{\alpha} \in C_{\gamma_k^{(\alpha)}} \cap [\xi, \alpha) \subseteq C$.
This shows that C is closed. □

Choose $\xi \leq \overline{\alpha} < \kappa$. Then there is $\alpha \in S$ with $\overline{\alpha} < \alpha \leq \gamma_k^{(\alpha)}$ and

$$C \cap \overline{\alpha} = (C \cap \alpha) \cap \overline{\alpha} = \left(C_{\gamma_k^{(\alpha)}} \cap [\xi, \alpha) \right) \cap \overline{\alpha} = C_{\gamma_k^{(\alpha)}} \cap [\xi, \overline{\alpha}).$$ □

Definition 5.5.13. Let $\vec{C} = (C_\alpha)_{\alpha < \kappa}$ be a C-sequence.

1. We say that $\vec{C}$ is *coherent* if for every $\alpha < \kappa$ and every limit point β of C_α, we have $C_\beta = C_\alpha \cap \beta$.
2. (Todorcevic) We say that $\vec{C}$ is a $\square(\kappa)$-sequence (Todorcevic-square-κ-sequence) if $\vec{C}$ is coherent and there is no closed unbounded subset C of κ such that $C_\alpha = C \cap \alpha$ for every limit point α of C.

Theorem 5.5.14 (Todorcevic)**.** *If $\vec{C}$ is a $\square(\kappa)$-sequence, then $\mathbb{T}(\rho_{\vec{C}})$ is a κ-Aronszajn tree.*

Proof. By Proposition 5.5.6, we know that $\mathrm{ht}(\mathbb{T}(\rho_{\vec{C}})) = \kappa$.

Lecture 24
14th Jan

Claim. *If $\alpha < \kappa$, then $\left|\mathbb{T}(\rho_{\vec{C}})(\alpha)\right| < \kappa$.*

Proof. By Lemma 5.5.11, it suffices to show that the set

$$A_\alpha = \{C_\beta \cap \alpha \mid \alpha \leq \beta < \kappa\}$$

has cardinality less than κ for every $\alpha \in \mathrm{Lim} \cap \kappa$.
Fix $\alpha \in \mathrm{Lim} \cap \kappa$ and assume that there is a sequence $(\beta_\delta)_{\delta<\kappa}$ of ordinals in $[\alpha, \kappa)$ and that $C_{\beta_\delta} \cap \alpha \neq C_{\beta_{\bar{\delta}}} \cap \alpha$ for all $\bar{\delta} < \delta < \kappa$. Define $r\colon \kappa \to \alpha + 1$ to be the unique function wit the following properties:

1. If $\alpha + 1$ contains no limit points of C_{β_δ}, then $r(\delta) = 0$.
2. Otherwise, let $r(\delta)$ denote the maximal limit point of C_{β_δ} in $\alpha + 1$.

Then there is $\lambda \leq \alpha$ and S unbounded in $\kappa \setminus \alpha$ such that r is constant on S with value λ. Since $(C_{\beta_\delta} \cap \alpha) \setminus r(\delta)$ is a finite subset of α and $[\alpha]^{<\omega}$ has cardinality less than κ, there are $\delta, \bar{\delta} \in S$ with $\bar{\delta} < \delta$ and $(C_{\beta_\delta} \cap \alpha) \setminus \lambda = \left(C_{\beta_{\bar{\delta}}} \cap \alpha\right) \setminus \lambda$. By the assumption, this implies that $\lambda \neq 0$ and λ is a limit point of C_{β_δ} and $C_{\beta_{\bar{\delta}}}$.
Since $\vec{C}$ is coherent, we can conclude that

$$\begin{aligned} C_{\beta_{\bar{\delta}}} \cap \alpha &= (C_{\beta_\delta} \cap \lambda) \cup (C_{\beta_\delta} \cap \alpha) \setminus \lambda = C_\lambda \cup (C_{\beta_\delta} \cap \alpha) \setminus \lambda \\ &= C_\lambda \cup \left(C_{\beta_{\bar{\delta}}} \cap \alpha\right) \setminus \lambda = \left(C_{\beta_{\bar{\delta}}} \cap \alpha\right) \setminus \lambda \\ &= C_{\beta_{\bar{\delta}}} \cap \alpha, \end{aligned}$$

contradiction. $\square$

Claim. $[\mathbb{T}(\rho_{\vec{C}})] = \emptyset$.

Proof. Assume hat $[\mathbb{T}(\rho_{\vec{C}})] \neq \emptyset$. By Lemma 5.5.12, there is a club C_* in κ and $\xi < \kappa$ such that for every $\xi \leq \alpha < \kappa$ there is $\alpha \leq \beta < \kappa$ with $C_* \cap \alpha = C_\beta \cap [\xi, \alpha)$. Let α_0 denote the least limit point of C_* and $C = C_* \cup C_{\alpha_0}$. Then there is $\alpha_0 \leq \beta < \kappa$ with $C_* \cap \alpha_0 = C_\beta \cap [\xi, \alpha_0)$. This implies that α_0 is a limit point of C_β and the coherence of $\vec{C}$ implies that $C_\beta \cap \alpha_0 = C_{\alpha_0}$ and $C_* \cap \alpha_0 = C_{\alpha_0} \cap [\xi, \alpha_0)$.
We can conclude that C is a club set in κ with $C \cap \alpha_0 = C_{\alpha_0}$. Let $\alpha > \alpha_0$ be a limit point of C. Then there is $\alpha \leq \beta < \kappa$ with $C_* \cap \alpha = C_\beta \cap [\xi, \alpha)$.
Since α is also a limit point of C_*, we know that α is also a limit point of C_β, $C_\alpha = C_\beta \cap \alpha$, α_0 is a limit point of C_α and $C_{\alpha_0} = C_\alpha \cap \alpha_0$.
This shows that

$$\begin{aligned} C \cap \alpha &= (C_* \cap \alpha) \cup C_{\alpha_0} = (C_\beta \cap [\xi, \alpha)) \cup C_{\alpha_0} \\ &= \left(C_\alpha \cap [\xi, \alpha)\right) \cup (C_\alpha \cap \alpha_0) = C_\alpha. \end{aligned}$$

This shows that $C_\alpha = C \cap \alpha$ holds for every limit point α of C and $\vec{C}$ is not a $\square(\kappa)$-sequence, contradiction. $\square$

This completes the proof of the theorem. $\square$

The following concept is of great importance in modern set theory.

Definition 5.5.15 (Jensen)**.** Let κ be an infinite cardinal and let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence. We say that $\vec{C}$ is a $\square_\kappa$-sequence (Jensen-Square-κ-sequence) if $\vec{C}$ is coherent and $\operatorname{type}(C_\alpha) \leq \kappa$ holds for all $\alpha < \kappa^+$.

Proposition 5.5.16. *Every $\square_\kappa$-sequence is a $\square(\kappa^+)$-sequence.*

Proof. Let $\vec{C} = (C_\alpha)_{\alpha<\kappa^+}$ be a $\square_\kappa$-sequence and assume that there is a club set C in κ^+ with $C_\alpha = C \cap \alpha$ for every limit point α of C.
Since κ^+ is regular, there is a limit point α' of C with

$$\kappa < \operatorname{type}(C \cap \alpha') = \operatorname{type}(C_{\alpha'}) \leq \kappa,$$

contradiction. $\square$

Proposition 5.5.17. *There is a $\square_\omega$-sequence.*

Proof. There is a sequence $\vec{C} = (C_\alpha)_{\alpha<\omega_1}$ with the following properties:

1. If $\alpha < \omega_1$, then $C_{\alpha+1} = \{\alpha\}$.
2. If $\alpha < \omega_1$ is a limit ordinal, then C_α is a cofinal subset of α with $\operatorname{type}(C_\alpha) = \omega$.

Then $\vec{C}$ is a $\square_\omega$-sequence. $\square$

Proof of Aronszajn's Theorem (Theorem 5.5.1). (Todorcevic)
By the previous proposition, there is a $\square_\omega$-sequence $\vec{C}$.
Then Proposition 5.5.16 shows that $\vec{C}$ is a $\square(\omega_1)$-sequence and Theorem 5.5.14 says that $\mathbb{T}(\rho_{\vec{C}})$ is an ω_1-Aronszajn tree. $\square$

Proof of Specker's Theorem (Theorem 5.5.2). (Todorcevic)
Let κ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$. In particular, κ is regular.
Then there is a C-sequence $\vec{C} = (C_\alpha)_{\alpha<\kappa^+}$ such that $\operatorname{type}(C_\alpha) = \operatorname{cof}(\alpha)$ holds for every limit ordinal $\alpha \in \mathrm{Lim} \cap \kappa^+$.

Claim. *If $\alpha < \kappa^+$, then $\left|\mathbb{T}(\rho_{\vec{C}})\right| \leq \kappa$.*

Proof. By the proof of Lemma 5.5.11, it suffices to show that the set

$$A_\alpha = \{C_\beta \cap \alpha \mid \alpha < \beta < \kappa^+\}$$

has cardinality at most κ for every $\alpha \in \mathrm{Lim}$. Since A_α consists of sets of order-type less than κ, the assumption implies $|A_\alpha| \leq |{}^{<\kappa}\alpha| \leq \kappa^{<\kappa} = \kappa$. $\square$

Claim. $[\mathbb{T}(\rho_{\vec{C}})] = \emptyset$.

Proof. Assume $[\mathbb{T}(\rho_{\vec{C}})] \neq \emptyset$. By Lemma 5.5.12, there is a club set C in κ^+ and $\xi < \kappa^+$ such that for every $\xi \leq \alpha < \kappa^+$ there is $\alpha \leq \beta < \kappa^+$ with $C \cap \alpha = C_\beta \cap [\xi, \alpha)$. Then $C \cap \xi = \emptyset$. There is a limit point α' of C with $\mathrm{type}(C \cap \alpha') > \kappa$.
Choose $\alpha' \leq \beta < \kappa^+$ with $C \cap \alpha' = C_\beta \cap [\xi, \alpha')$.
Then $\mathrm{type}(C_\beta) \geq \mathrm{type}\left(C_\beta \cap [\xi, \alpha')\right) = \mathrm{type}(C \cap \alpha') > \kappa$, contradiction. □

These claims show that $\mathbb{T}(\rho_{\vec{C}})$ is a κ^+-Aronszajn tree. □

5.6 Higher Partition Relations and Weakly Compact Cardinals

By Ramsey's Theorem (Theorem 5.3.7), we have $\omega \to (\omega)^n_m$ for all $0 < m, n < \omega$. In the following section, we will consider generalisations of this statement to higher cardinalities. The next theorem is a corollary of the Ramification Lemma (Theorem 5.3.6).

Definition 5.6.1. Let κ be an infinite cardinal. We define along the ordinals the $\beth$-sequence (Bet-sequence) of κ

$$\left(\beth_\alpha(\kappa)\right)_{\alpha \in \mathrm{Ord}}$$

by the following clauses:

1. $\beth_0(\kappa) = \kappa$.
2. $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$.
3. $\beth_\alpha(\kappa) = \sup\{\beth_{\overline{\alpha}}(\kappa) \mid \overline{\alpha} < \alpha\}$ for all $\alpha \in \mathrm{Lim}$.

Theorem 5.6.2 (Erdős-Rado). *If κ is an infinite cardinal an $n < \omega$, then*

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa.$$

Proof. We prove the statement by induction on $n < \omega$. The case $n = 0$ follows from the regularity of κ^+. Assume that the statement holds for $n < \omega$ and let $\nu = \beth_{n+1}(\kappa)^+$ and $c\colon [\nu]^{n+2} \to \kappa$ be a function.
Let $\mathbb{T}_c = (\nu, \leq_c)$ be the tree given by the Ramification Lemma (Theorem 5.3.6).
Then we have

$$|\mathbb{T}_c(\alpha)| \leq \kappa^{|\alpha+\omega|} \leq \kappa^{\beth_n(\kappa)} = \beth_{n+1}(\kappa) < \nu$$

for all $\alpha < \beth_n(\kappa)^+$. This implies that $\mathbb{T}_c(\beth_n(\kappa)^+) \neq \emptyset$, because otherwise $\nu \leq \beth_n(\kappa)^+ \cdot \beth_{n+1}(\kappa) < \nu$. Choose $\eta \in \mathbb{T}_c(\beth_n(\kappa)^+)$ and let $C = \mathrm{pred}_{\mathbb{T}_c}(\eta)$. By the Ramification Lemma (Theorem 5.3.6), there is a function $d\colon [C]^{n+1} \to \kappa$ such that

$$d(\{\alpha_0, \ldots, \alpha_n\}) = c(\{\alpha_0, \ldots, \alpha_n, \beta_0\}) = c(\{\alpha_0, \ldots, \alpha_n, \beta_1\})$$

for all $\alpha_0, \ldots, \alpha_n, \beta_0, \beta_1 \in C$ with $\alpha_0 < \ldots < \alpha_n < \beta_0 < \beta_1$.
Since C has order-type $\beth_n(\kappa)^+$, the inductive assumption yields $H \in [C]^\kappa$ that is homogeneous for d. Then H is also homogeneous for c. □

We will now show that the above result is optimal in a certain sense. Lecture 25 19th Jan

Proposition 5.6.3. *If κ is an infinite cardinal, then $2^\kappa \nrightarrow (3)^2_\kappa$.*

Proof. Let $(x_\gamma)_{\gamma<2^\kappa}$ be an injective enumeration of the set ${}^2\kappa$ of all functions from κ to 2. Given $x, y \in {}^2\kappa$ with $x \neq y$, we define

$$\Delta(x, y) = \min\{\alpha < \kappa \mid x(\alpha) \neq y(\alpha)\}.$$

If we define $c\colon [2^\kappa]^2 \to \kappa, \{\gamma, \delta\} \mapsto \Delta(x_\gamma, x_\delta)$, then there is no $H \in [2^\kappa]^3$ that is homogeneous for c. □

Lemma 5.6.4 (Kurepa-Sierpiński)**.** *If κ is an infinite cardinal, then $2^\kappa \nrightarrow (\kappa^+)^2_2$.*

Proof. Fix an injective enumeration $(x_\gamma)_{\gamma<2^\kappa}$ of ${}^2\kappa$ and let

$$x\colon [2^\kappa]^2 \to 2$$

denote the unique function with

$$c(\{\gamma, \delta\}) = 0 \text{ iff } x_\gamma(\Delta(x_\gamma, x_\delta)) < x_\delta(\Delta(x_\gamma, x_\delta)), \text{ for all } \gamma < \delta < 2^\kappa.$$

Assume that there is $H \in [2^\kappa]^{\kappa^+}$ that is homogeneous for c.

1. Assume that $c{\restriction}[H]^2$ is constant with value 0.
   Then we have

   $$\gamma < \delta \text{ iff } x_\gamma(\Delta(x_\gamma, x_\delta)) < x_\delta(\Delta(x_\gamma, x_\delta)), \text{ for all } \gamma, \delta \in H \text{ with } \gamma \neq \delta.$$

   Given $\gamma \in H$, let $r(\gamma)$ denote the minimal $\alpha < \kappa$ such that $\alpha = \Delta(x_\gamma, x_\delta)$ and $x_\gamma(\alpha) < x_\delta(\alpha)$ for some $\delta \in H$. We define

   $$H_\alpha = \{\gamma \in H \mid r(\gamma) = \alpha\}, \text{ for } \alpha < \kappa.$$

   Claim. *If $\alpha < \kappa, \gamma_0 \in H_\alpha$ and $\delta \in H$ with $\alpha = \Delta(x_{\gamma_0}, x_\delta)$ and $x_{\gamma_0}(\alpha) < x_\delta(\alpha)$, then $H_\alpha \subseteq \delta$. In particular, we have $|H_\alpha| \leq \kappa$ for all $\alpha < \kappa$.*

   Proof. By assumption, we have $\gamma_0 < \delta$. Choose $\gamma_1 \in H_\alpha$ with $\gamma_0 \neq \gamma_1$.
   Then $x_{\gamma_0}{\restriction}\alpha = x_{\gamma_1}{\restriction}\alpha$, because otherwise there would be $i < \alpha$ with $\gamma_i \in H_{\overline{\alpha}}$ for some $\overline{\alpha} < \alpha$, contadicting $\gamma_0, \gamma_1 \in H_\alpha$. Since there is $\overline{\delta} \in H$ with $x_{\gamma_1}(\alpha) < x_{\overline{\delta}}(\alpha)$, we know that $0 = x_{\gamma_0}(\alpha) = x_{\gamma_1}(\alpha) < x_\delta(\alpha) = 1$. By the above equivalence, this implies that $\gamma_1 < \delta$.
   The second part of the claim follows from $\mathrm{type}(H \cap \delta) < \mathrm{type}(H) = \kappa^+$. □

   Since $H = \bigcup_{\alpha<\kappa} H_\alpha$, the above claim yields a contradiction.

2. Assume now that $c\restriction[H]^2$ is constant with value 1.
   Then we have

$$\gamma < \delta \text{ iff } x_\delta(\Delta(x_\gamma, x_\delta)) < x_\gamma(\Delta(x_\gamma, x_\delta)), \text{ for all } \gamma, \delta \in H \text{ with } \gamma \neq \delta.$$

   In this case, we can run an analogous argument to derive a contradiction.

□

We will now consider direct generalisations of Ramsey's Theorem (Theorem 5.3.7) to uncountable cardinals.

Definition 5.6.5. An uncountable cardinal κ is called *weakly compact* if $\kappa \to (\kappa)^2_2$ holds.

Lemma 5.6.6 (Erdős)**.** *Weakly compact cardinals are inaccessible.*

Proof. Let κ be weakly compact.

1. Assume κ is singular.
   Then there is $\lambda < \kappa$ and a sequence $(A_\alpha)_{\alpha<\lambda}$ of disjoint subsets of κ with cardinality less than κ such that $\kappa = \bigcup_{\alpha<\lambda} A_\alpha$.
   Let $c\colon [\kappa]^2 \to 2$ be the unique function with

$$c(\{\gamma, \delta\}) = 0 \text{ iff there is } \alpha < \lambda \text{ with } \gamma, \delta \in A_\alpha.$$

   Then there is $H \in [\kappa]^\kappa$ homogeneous for c. We know that $c\restriction[H]^2$ is not constant with value 0, because otherwise $H \subseteq A_\alpha$ for some $\alpha < \lambda$. Hence, $c\restriction[H]^2$ is constant with value 1 and there is an injection $i\colon H \to \lambda$ with $\gamma \in A_{i(\gamma)}$ for all $\gamma \in H$, contradiction. Thus, κ is regular.

2. Now assume that κ is not a strong limit cardinal, i.e there is $\eta < \kappa$ with $2^\eta \geq \kappa$. By Lemma 5.3.4, we get $2^\eta \to (\eta^+)^2_2$ and this contradicts Lemma 5.6.4.
   Therefore, κ is a strong limit cardinal.

Hence, κ is inaccessible. □

The next result characterises weak compactness with the help of Aronszajn trees.

Theorem 5.6.7 (Erdős-Rado-Scott-Tarski)**.** *Let κ be an uncountable cardinal. Then the following statements are equivalent:*

1. *κ is weakly compact.*

2. *$\kappa \to (\kappa)^n_\lambda$ for all $0 < n < \omega, 0 < \lambda < \kappa$.*

3. *κ is inaccessible and has the tree property.*

Proof.

(2) $\Rightarrow$ (1): Let $n = 2, \lambda = 2$.

(1) ⇒ (3): Assume κ is weakly compact. By the previous lemma, κ is inaccessible.
Assume that there is a κ-Aronszajn tree $\mathbb{T}$. Since $\mathrm{ht}(\mathbb{T}) = \kappa$ and $[\mathbb{T}] = \emptyset$, there is a sequence $(b_\gamma)_{\gamma<\kappa}, b_\gamma \in \partial\mathbb{T}$ for all $\gamma < \kappa$, such that $\mathrm{lh}_{\mathbb{T}}(b_{\overline{\gamma}}) < \mathrm{lh}_{\mathbb{T}}(b_\gamma)$ for all $\overline{\gamma} < \gamma < \kappa$.
Let $\preceq$ be a suitable linear ordering of $\mathbb{T}$ and let $\preceq_{\mathrm{lex}}$ denote the resulting linear ordering of $\partial\mathbb{T}$ (see Definition 5.4.10).
Let $c\colon [\kappa]^2 \to 2$ be the unique function with

$$c(\{\gamma,\delta\}) = 0 \text{ iff } b_\gamma \prec_{\mathrm{lex}} b_\delta \text{ for all } \gamma < \delta < \kappa.$$

By assumption, there is $H \in [\kappa]^\kappa$ homogeneous for c.
Assume that $c{\restriction}[H]^2$ is constant with value 0 (the case for the constant value 1 is analogous).
Given $\gamma, \delta \in H$ with $\gamma \neq \delta$, this implies that $\gamma < \delta$ if and only if $b_\gamma \prec_{\mathrm{lex}} b_\delta$.

Claim. *If $\alpha < \kappa$, then there are $t_\alpha \in \mathbb{T}(\alpha)$ and $\alpha < \xi_\alpha < \kappa$ such that $t_\alpha = b_\gamma(\alpha)$ for all $\xi_\alpha < \gamma < \kappa$ with $\gamma \in H$.*

Proof. Since $|\mathbb{T}(\alpha)| < \kappa$, there is $t \in \mathbb{T}(\alpha)$ with $b_\gamma(\alpha) = t$ for unboundedly many $\gamma \in H$. Let $\alpha < \xi \in H$ be minimal with $t \in b_\xi(\alpha)$.
Choose $\gamma, \overline{\gamma} \in H$ with $\xi < \overline{\gamma} < \gamma$ and $b_\gamma(\alpha) = t$.
We then have $b_\xi \prec_{\mathrm{lex}} b_{\overline{\gamma}} \prec_{\mathrm{lex}} b_\gamma$ and $b_\gamma(\alpha) = b_\xi(\alpha)$ implies that $\Delta(b_\xi, b_\gamma) > \alpha$.
If

$$\delta = \Delta(b_\xi, b_{\overline{\gamma}}) \leq \min\big(\alpha, \Delta(b_{\overline{\gamma}}, b_\gamma)\big), \text{ then } b_\xi(\delta) \prec b_{\overline{\gamma}}(\delta) \preceq b_\gamma(\delta) = b_\xi(\alpha),$$

contradiction. If

$$\delta = \Delta(b_{\overline{\gamma}}, b_\gamma) \leq \min\big(\alpha, \Delta(b_\xi, b_{\overline{\gamma}}\big), \text{ then } b_\xi(\delta) \preceq b_{\overline{\gamma}}(\delta) \prec b_\gamma(\delta) = b_\xi(\delta),$$

contradiction. This shows that $\alpha < \min\big(\Delta(b_\xi, b_{\overline{\gamma}}), \Delta(b_{\overline{\gamma}}, b_\gamma)\big)$, hence $b_{\overline{\gamma}}(\alpha) = t$. □

If $\overline{\alpha} < \alpha < \kappa$, then there is $\max(\xi_\alpha, \xi(\overline{\alpha})) < \gamma < \kappa$ with $\gamma \in H$ and we have $t_{\overline{\alpha}} = b_\gamma(\overline{\alpha}) <_{\mathbb{T}} b_\gamma(\alpha) = t_\alpha$. This shows that $\{t_\alpha \mid \alpha < \kappa\} \in [\mathbb{T}] = \emptyset$, contradiction.

(3) ⇒ (2): Assume now that κ is an inaccessible cardinal with the tree property.
Given $0 < \lambda < \kappa$, we prove $\kappa \to (\kappa)^{n+1}_\lambda$ by induction on $n < \omega$. The case $n = 0$ follows from the regularity of κ. Assume that $\kappa \to (\kappa)^{n+1}_\lambda$ and let $c\colon [\kappa]^{n+2} \to \lambda$ be a function. Let $\mathbb{T}_c = (\kappa, \leq_c)$ denote the tree given by the Ramification Lemma (Theorem 5.3.6).
Then $|\mathbb{T}_c(\alpha)| \leq \lambda^{|\alpha+\omega|} < \kappa$ for all $\alpha < \kappa$ (since κ is inaccessible) and this implies that $\mathrm{ht}(\mathbb{T}_c) = \kappa$. Since $\mathbb{T}_c$ is not a κ-Aronszajn tree, there is some $b \in [\mathbb{T}_c]$. By the properties of $\mathbb{T}_c$, there is a function $d\colon [b]^{n+1} \to \lambda$ with

$$d(\{\alpha_0, \ldots, \alpha_n\}) = c(\{\alpha_0, \ldots, \alpha_n, \beta_0\}) = c(\{\alpha_0, \ldots, \alpha_n, \beta_1\})$$

for all $\alpha_0, \ldots, \alpha_n, \beta_0, \beta_1 \in b$ with $\alpha_0 < \ldots < \alpha_n < \beta_0 < \beta_1$.
The induction hypothesis implies that there is an $H \in [b]^\kappa$ that is homogeneous for d and it follows that H is also homogeneous for c.

□

Remark 5.6.8. We will later use the above theorem to show that if there are weakly compact cardinals, then the least inaccessible cardinal is smaller than the least weakly compact cardinal.

Corollary 5.6.9. *If κ is weakly compact, then there is no $\square(\kappa)$-sequence.*

Remark 5.6.10. Profound results of Jensen show that in certain important models of ZFC, the non-existence of $\square(\kappa)$-sequences characterises weak compactness.

5.7 Special Aronszajn Trees and Mahlo Cardinals

In this section, we will use the following notion to show that there always is an inaccessible cardinal below a weakly compact cardinal, if the latter exists. Lecture 26 21st Jan

Definition 5.7.1. Given an inaccessible cardinal κ, we say that κ is a *Mahlo cardinal* if the set of regular cardinals smaller than κ is stationary in κ, i.e. if every club subsetet of κ contains some regular cardinals.

Proposition 5.7.2. *Let κ be an uncountable cardinal. Then the following statements are equivalent:*

1. *κ is Mahlo.*
2. *The set of inaccessible cardinals smaller than κ is stationary in κ.*

Proof.

(1) ⇒ (2): Assume that κ is Mahlo. Since κ is inaccessible, there is a club $C \subseteq \kappa$ consisting of strong limit cardinals. This is due to the fact that every strong limit cardinal with uncountable cofinality there is a minimal $\alpha \in \mathrm{Ord}$ with $\beth_\alpha \geq \kappa$ (Remember that the $\beth$-numbers were defined as $\beth_0 = \aleph_0, \beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\alpha = \sup_{\beta<\alpha} \beth_\beta$ for $\alpha \in \mathrm{Lim}$). Then $\kappa = \beth_\alpha$ and $\mathrm{cof}(\alpha) > \omega$. $C = \{\beth_{\overline{\alpha}} \mid \overline{\alpha} \in \mathrm{Lim} \cap \alpha\}$ is a club set in κ that consists of strong limit cardinals. Let S be the set of regular cardinals less than κ. By assumption, S is stationary in κ, so $S \cap C$ is also stationary in κ and this set consists of inaccessible cardinals.

(2) ⇒ (1): Assume that the set of inaccessible cardinals below κ is stationary in κ. Then κ is a limit of inaccesible cardinals. Hence κ is a strong limit cardinal, and, by assumption, κ is inaccessible.
Since every inaccessible cardinal is regular, we know that the set of regular cardinals less than κ is stationary in κ, hence κ is Mahlo.

□

We will characterise Mahlo cardinals in the following by the non-existence of certain Aronszajn trees.

Definition 5.7.3 (Todorcevic)**.** Let κ be an uncountable regular cardinal and $\mathbb{T}$ be a tree of height κ.

1. A function $r\colon \mathbb{T} \to \mathbb{T}$ is called *regressive* if $r(t) <_{\mathbb{T}} t$ holds for every $t \in \mathbb{T}\setminus\{\mathrm{root}(\mathbb{T})\}$.
2. The tree $\mathbb{T}$ is called *special* if there is a regressive map $r\colon \mathbb{T} \to \mathbb{T}$ with the property that $r^{-1}[\{t\}]$ is the union of less than κ-many antichains in $\mathbb{T}$, for every $t \in \mathbb{T}$. In other words, for every $t \in \mathbb{T}$ there is some $\lambda < \kappa$ and a function $c_t\colon r^{-1}[\{t\}] \to \lambda$ such that $c_t(s_0) \neq c_t(s_1)$ for all $s_0, s_1 \in \mathbb{T}$ with $r(s_0) = r(s_1) = t$ and $s_0 \leq_{\mathbb{T}} s_1$.

Lemma 5.7.4. *Let κ be an uncountable regular cardinal and let $\mathbb{T}$ be a tree of height κ. If $\mathbb{T}$ is special, then $[\mathbb{T}] = \emptyset$.*

Proof. Assume that there is $b \in [\mathbb{T}]$ and let $r\colon \mathbb{T} \to \mathbb{T}$ be the regressive function witnessing that $\mathbb{T}$ is special. Then there is a stationary subset $S \subseteq \kappa$ and $\alpha < \kappa$ such that $r(b(\beta)) = b(\alpha)$ for all $\beta \in S$.
Choose a function $c\colon r^{-1}[\{b(\alpha)\}] \to \lambda$ with $\lambda < \kappa$ and such that $c(t_0) \neq c(t_1)$ holds for $t_0, t_1 \in r^{-1}[\{b(\alpha)\}]$ with $t_0 \leq_{\mathbb{T}} t_1$ – we will refer to such functions as being *injective on chains*. But then there are $\beta_0, \beta_1 \in S$ with $\beta_0 \neq \beta_1$, and $c(b(\beta_0)) = c(b(\beta_1))$, contradiction. □

Lemma 5.7.5. *Let κ be an uncountable regular cardinal and let $\mathbb{T}$ be a κ-Suslin tree. Then $\mathbb{T}$ is not special.*

Proof. Assume that $\mathbb{T}$ is special and let $r_0\colon \mathbb{T} \to \mathbb{T}$ be a regressive map that witnesses that $\mathbb{T}$ is special. Given $\alpha \in \mathrm{Lim} \cap \kappa$, choose $t_\alpha \in \mathbb{T}(\alpha)$ and set $r(\alpha) = \mathrm{lh}_{\mathbb{T}}(r_0(t_\alpha)) < \alpha$. This defines a regressive function $r\colon \mathrm{Lim} \cap \kappa \to \kappa$ and, by Fodor's Lemma (Theorem 4.1.9), there is $S_0 \subseteq \mathrm{Lim}$ stationary in κ and $\lambda < \kappa$ with $r(\alpha) = \lambda$ for all $\alpha \in S_0$.
By Proposition 5.4.8, we know that $\mathbb{T}$ is a κ-Aronszajn tree and therefore we have $|\mathbb{T}(\lambda)| < \kappa$ and, again by Fodor's Lemma, there is $S \subseteq S_0$ stationary in κ ad $t \in \mathbb{T}(\lambda)$ such that $r(t_\alpha) = t$ for all $\alpha \in S$. Then $r^{-1}[\{t\}]$ has cardinality κ and this set is not the union of less than κ-many antichains in $\mathbb{T}$, because the assumption implies that all such antichains have cardinality less than κ.
Hence r_0 does not witness that $\mathbb{T}$ is special, contradiction. □

The following result shows that special trees of successor heights can be characterised by a simpler criterion.

Theorem 5.7.6 (Todorcevic)**.** *Let κ be an infinite cardinal and let $\mathbb{T}$ be a tree of height κ^+. Then the following statements are equivalent:*

1. *$\mathbb{T}$ is special.*
2. *$\mathbb{T}$ is the union of less than κ-many antichains in $\mathbb{T}$. In other words, there is a function $c\colon \mathbb{T} \to \kappa$ that is injective on chains (see proof of Lemma 5.7.4 for the definition) in $\mathbb{T}$.*

Remark 5.7.7. A theorem of Baumgartner-Mulitz-Reinhardt shows that the consistency of ZFC implies the consistency of ZFC + "all Aronszajn trees are special" (see the Models of Set Theory II course). By Lemma 5.7.5, the Suslin Hypothesis holds in models of this theory.

Remark 5.7.8. The label '5.7.8" is omitted to keep the numbering in accordance with the lecture.

The following theorem characterises Mahlo cardinals using special Aronszajn trees.

Theorem 5.7.9 (Todorcevic). *Let κ be an inaccessible cardinal. The following statements are equivalent:*

1. *κ is Mahlo.*
2. *There are no special κ-Aronszajn trees.*

Corollary 5.7.10 (Hanf). *Every weakly compact cardinal is a Mahlo cardinal.*

Proof. Let κ be a weakly compact cardinal.
By Lemma 5.6.6, κ is inaccessible and, by Theorem 5.6.7, κ also has the tree property. Hence, there are no κ-Aronszajn trees and, in particular, no special κ-Aronszajn trees. By Theorem 5.7.9, κ is Mahlo. □

Proof of Theorem 5.7.6 Let $r\colon \mathbb{T} \to \mathbb{T}$ be a regressive function that witnesses that $\mathbb{T}$ is special. Lecture 27 26th Jan
Given $t \in \mathbb{T}$, there is a map $c_t\colon r^{-1}[\{t\}] \to \kappa$ that is injective on chains (see proof of Lemma 5.7.4 for the definition). Fix a surjection $\varphi\colon \kappa \to \kappa \times \kappa \times \kappa$ such that $\alpha_0 \leq \alpha$ holds for all $\alpha < \kappa$ with $\varphi(\alpha) = (\alpha_0, \alpha_1, \alpha_2)$.
Given $t \in \mathbb{T} \setminus \{\mathrm{root}(\mathbb{T})\}$, we inductively construct a $\leq_{\mathbb{T}}$-increasing sequence $\big(u_t(\alpha)\big)_{\alpha<\kappa}$ in $\mathrm{pred}_{\mathbb{T}}(t) \cup \{t\}$ such that the following statements hold for all $\alpha < \kappa$:

1. $u_t(0) = \mathrm{root}(\mathbb{T})$.
2. If $\alpha \in \mathrm{Lim}$, then $u_t(\alpha)$ is the supremum of $\{u_t(\overline{\alpha}) \mid \overline{\alpha} < \alpha\}$ in $\mathrm{pred}_{\mathbb{T}}(t) \cup \{t\}$.
3. $u_t(\alpha) = u_t(\alpha + 1)$ if and only if $u_t(\alpha) = t$.
4. Assume that $u_t(\alpha) <_{\mathbb{T}} t$ and $\varphi(\alpha) = (\alpha_0, \alpha_1, \alpha_2)$ with $\alpha_0 \leq \alpha$. If there is some $u \in \mathbb{T}(\alpha_2)$ such that $u \leq_{\mathbb{T}} u_t(\alpha_0)$ and the set
$$c_u^{-1}[\{\alpha_1\}] \cap \{s \in \mathbb{T} \mid u_t(\alpha) <_{\mathbb{T}} s \leq_{\mathbb{T}} t\}$$
is non-empty, then $u_t(\alpha + 1)$ is the unique element in this set.
Otherwise, $u_t(\alpha + 1)$ is the $\leq_{\mathbb{T}}$-minimal element of
$$\{s \in \mathbb{T} \mid u_t(\alpha) <_{\mathbb{T}} s \leq_{\mathbb{T}} t\}.$$

Claim. *If $t \in \mathbb{T} \setminus \{\mathrm{root}(\mathbb{T})\}$, then there is a minimal $\alpha_t < \kappa$ with $u_t(\alpha_t) = t$.*

Proof. Assume that $u_t(\alpha) <_{\mathbb{T}} t$ holds for all $\alpha < \kappa$ and define $\bar{t}$ to be the supremum of $\{u_t(\alpha) \mid \alpha < \kappa\}$ in $\mathrm{pred}_{\mathbb{T}}(t) \cup \{t\}$.
Set $u = r(\bar{t}), \alpha_2 = \mathrm{lh}_{\mathbb{T}}(u)$ and $\alpha_1 = c_u(\bar{t})$.
Choose $\alpha_0 < \kappa$ with $u \leq_{\mathbb{T}} u_t(\alpha_0)$ and $\alpha < \kappa$ with $\varphi(\alpha) = (\alpha_0, \alpha_1, \alpha_2)$ and $\alpha_0 \leq \alpha$. Then $u \in \mathbb{T}(\alpha_2), u \leq_{\mathbb{T}} u_t(\alpha_0)$ and

$$\bar{t} \in c_u^{-1}[\{\alpha_1\}] \cap \{s \in \mathbb{T} \mid u_t(\alpha) <_{\mathbb{T}} s \leq_{\mathbb{T}} t\}.$$

By definition, this implies that $\bar{t} = u_t(\alpha + 1)$, contradiction. □

Given $\alpha < \kappa$, we define

$$X_\alpha = \{t \in \mathbb{T} \setminus \{\mathrm{root}(\mathbb{T})\} \mid \alpha_t = \alpha\}.$$

Claim. *If $\alpha < \kappa$, then X_α contains no chain of order-type $|\alpha|^+$.*

Proof. Assume that there is a chain c in $\mathbb{T}$ mit $c \subseteq X_{\alpha_*}$ and $\mathrm{lh}_{\mathbb{T}}(c) = \theta = |\alpha_*|^+$. By induction on $\alpha \leq \alpha_*$, we show that the sequences $\big(u_{c(\gamma)}(\alpha)\big)_{\gamma<\theta}$ are eventually constant. The case $\alpha = 0$ is clear and the case $\alpha \in \mathrm{Lim}$ follows directly from the induction hypothesis using $\mathrm{cof}(\alpha) < \theta$.
Fix $\alpha \leq \alpha_*$ and assume that there is some $\gamma_* < \theta$ with $u_{c(\gamma)}(\alpha) = u_{c(\gamma_*)}(\alpha)$ for all $\gamma_* \leq \gamma \leq \theta$. Let $\varphi(\alpha) = (\alpha_0, \alpha_1, \alpha_2)$.

1. Assume that there is $\gamma_* \leq \overline{\gamma} < \theta$ and $u \in \mathbb{T}(\alpha_2)$ such that $u \leq_{\mathbb{T}} u_{c(\overline{\gamma})}(\alpha_0)$ and

   $$c_u^{-1}[\{\alpha_1\}] \cap \{s \in \mathbb{T} \mid u_{c(\overline{\gamma})}(\alpha) <_{\mathbb{T}} s \leq_{\mathbb{T}} c(\overline{\gamma})\}$$

   is non-empty.
   Then the unique element of this intersection is equal to $u_{c(\gamma)}(\alpha + 1)$ for every $\overline{\gamma} \leq \gamma < \theta$. Hence the sequence is eventually constant.

2. Assume that for every $\gamma_* \leq \gamma < \theta$, we have either $u_{c(\gamma)}(\alpha) = u_{c(\gamma)}(\alpha + 1)$ or $u_{c(\gamma)}(\alpha + 1)$ is the minimal element of $\{s \in \mathbb{T} \mid u_{c(\gamma)}(\alpha) <_{\mathbb{T}} s \leq_{\mathbb{T}} c(\gamma)\}$. Then the sequence is also eventually constant.

□

This shows that $\mathrm{lh}_{\mathbb{T}}(\mathrm{pred}_{\mathbb{T}}(t) \cap X_{\alpha_t}) < \kappa$ for every $t \in \mathbb{T} \setminus \{\mathrm{root}(\mathbb{T})\}$. We can conclude that the function

$$c \colon \mathbb{T} \setminus \{\mathrm{root}(\mathbb{T})\} \to \kappa \times \kappa, \quad t \mapsto \big(\alpha_t, \mathrm{lh}_{\mathbb{T}}(\mathrm{pred}_{\mathbb{T}}(t) \cap X_{\alpha_t})\big)$$

is injective on chains in $\mathbb{T}$. □

Proof of Theorem 5.7.9, (1) implies (2). Let κ be a Mahlo cardinal and let $\mathbb{T}$ be a κ-Aronszajn tree.
Assume that there is a regressive function $r \colon \mathbb{T} \to \mathbb{T}$ witnessing that $\mathbb{T}$ is special. Let

I denote the set of inaccessible cardinals less than κ. $\delta \in I$, the tree $\mathbb{T}_{<\delta} = (\mathbb{T}_{<\delta}, \leq_{\mathbb{T}})$ with $\mathbb{T}_{<\delta} = \{t \in \mathbb{T} \mid \operatorname{rnk}_{\mathbb{T}}(t) < \delta\}$ (for the definition of $\operatorname{rnk}_{\mathbb{T}}$, see Problem 41), which is an initial segment of $\mathbb{T}$, has height δ.
By Lemma 5.7.4, $\mathbb{T}_{<\delta}$ is not special, because every element of $\mathbb{T}(\delta)$ induces a cofinal branch through $\mathbb{T}_{<\delta}$ and δ is uncountable and regular.
Given $\delta \in I$, there is some $t_\delta \in \mathbb{T}_{<\delta}$ such that $(r{\restriction}\mathbb{T}_{<\delta})^{-1}[\{t_\delta\}]$ is not the union of less than δ-many antichains in $\mathbb{T}_{<\delta}$.
By Fodor's Lemma (Theorem 4.1.9) and the assumption, there is $S_0 \subseteq I$ stationary in κ and $\alpha < \kappa$ such that $t_\delta \in \mathbb{T}(\alpha)$ for all $\delta \in S_0$.
Since $|\mathbb{T}(\alpha)| < \kappa$, another application of Fodor's Lemma yields $S \subseteq S_0$ stationary in κ and some $t \in \mathbb{T}(\alpha)$ such that $t = t_\delta$ for all $\delta \in S$. Then there is $\lambda < \kappa$ and a sequence $(A_\gamma)_{\gamma<\lambda}$ of antichains in $\mathbb{T}$ such that $r^{-1}[\{t\}] = \bigcup_{\gamma<\lambda} A_\gamma$.
Choose $\lambda < \delta \in S$. Then

$$(r{\restriction}\mathbb{T}_{<\delta})^{-1}[\{t_\delta\}] = \bigcup_{\gamma<\lambda} A_\gamma \cap \mathbb{T}_{<\delta}$$

is the union of less than δ-many antichains in $\mathbb{T}_{<\delta}$, contradiction. □

To prove the converse implication, we use the notion of walks on ordinals.

Lemma 5.7.11. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence and $\alpha < \overline{\beta} < \beta < \kappa$ with $\alpha \in \mathrm{Lim}$ and $\rho_{\vec{C}}(\cdot,\beta){\restriction}\alpha = \rho_{\vec{C}}(\cdot,\overline{\beta}){\restriction}\alpha$:*
Let $(\gamma_0,\ldots,\gamma_n)$ denote the walk from β to α through $\vec{C}$ and let $(\overline{\gamma}_0,\ldots,\overline{\gamma}_m)$ denote the walk from $\overline{\beta}$ to α through $\vec{C}$. Assume that either

$$\sup\left(C_{\gamma_{n-1}} \cap \alpha\right), \sup\left(C_{\overline{\gamma}_{m-1}} \cap \alpha\right) < \alpha$$

or

$$\sup\left(C_{\gamma_{n-1}} \cap \alpha\right) = \sup\left(C_{\overline{\gamma}_{m-1}} \cap \alpha\right) = \alpha.$$

Then $\rho_{\vec{C}}(\alpha,\beta) = \rho_{\vec{C}}(\alpha,\overline{\beta})$.

Lemma 5.7.12. *Let $\vec{C} = (C_\alpha)_{\alpha<\kappa}$ be a C-sequence and $\overline{\alpha} < \alpha < \beta < \kappa$.*
Then $\rho_{\vec{C}}(\alpha,\beta) = \rho_{\vec{C}}(\overline{\alpha},\beta)$.

Proof. Let $(\gamma_0,\ldots,\gamma_n)$ denote the walk from β to α through $\vec{C}$ and let $(\overline{\gamma}_0,\ldots,\overline{\gamma}_m)$ denote the walk from β to $\overline{\alpha}$ through $\vec{C}$.
If α appears in $(\overline{\gamma}_0,\ldots,\overline{\gamma}_m)$, then Lemma 5.5.9 implies that $\rho_{\vec{C}}(\alpha,\beta)$ is a proper initial segment of $\rho_{\vec{C}}(\overline{\alpha},\beta)$ and hence $\rho_{\vec{C}}(\alpha,\beta) \neq \rho_{\vec{C}}(\overline{\alpha},\beta)$.
Otherwise, let $k > 0$ be minimal with the property that $\gamma_k \neq \overline{\gamma}_k$.
We have $\gamma_k = \min\left(C_{\gamma_{k-1}} \cap \alpha\right)$ and $\overline{\gamma}_k = \min\left(C_{\gamma_{k-1}} \cap \overline{\alpha}\right)$.
Since $\overline{\alpha} < \alpha$, this implies that $C_{\gamma_{k-1}} \cap \overline{\alpha} \subsetneq C_{\gamma_{k-1}} \cap \alpha$ and
$\operatorname{type}\left(C_{\gamma_{k-1}} \cap \overline{\alpha}\right) < \operatorname{type}\left(C_{\gamma_{k-1}} \cap \alpha\right)$. We can conclude that

$$\rho_{\vec{C}}(\alpha,\beta)(k-1) = \operatorname{type}\left(C_{\gamma_{k-1}} \cap \alpha\right) \neq \operatorname{type}\left(C_{\gamma_{k-1}} \cap \overline{\alpha}\right) = \rho_{\vec{C}}(\overline{\alpha},\beta)(k-1).$$

□

Proof of Lemma 5.7.11. Note that $\sup\left(C_{\gamma_{n-1}} \cap \alpha\right), \sup\left(C_{\overline{\gamma}_{m-1}} \cap \alpha\right)$ can only be unbounded in α in the last or second-to-last step. Lecture 28 28th Jan

1. Assume
$$\sup\left(C_{\gamma_{n-1}} \cap \alpha\right), \sup\left(C_{\overline{\gamma}_{m-1}} \cap \alpha\right) < \alpha.$$
Then $C_{\gamma_i} \cap \alpha$ is bounded in α for all $i < n$ and $C_{\overline{\gamma}_i} \cap \alpha$ is bounded in α for all $i < m$. By Lemma 5.5.8, there is $\xi < \alpha$ such that for all $\overline{\alpha} \in [\xi, \alpha)$, we know that α appears in the walk from β to $\overline{\alpha}$ through $\vec{C}$. By Lemma 5.5.9, the assumption that $\rho_{\vec{C}}(\cdot, \beta) \restriction \alpha = \rho_{\vec{C}}(\cdot, \overline{\beta}) \restriction \alpha$ implies that
$$\rho_{\vec{C}}(\alpha, \beta) ^\frown \rho_{\vec{C}}(\overline{\alpha}, \alpha) = \rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta}) = \rho_{\vec{C}}(\alpha, \overline{\beta}) ^\frown \rho_{\vec{C}}(\overline{\alpha}, \alpha),$$
and hence $\rho_{\vec{C}}(\alpha, \beta) = \rho_{\vec{C}}(\alpha, \overline{\beta})$.

2. Assume
$$\sup\left(C_{\gamma_{n-1}} \cap \alpha\right) = \sup\left(C_{\overline{\gamma}_{m-1}} \cap \alpha\right) = \alpha.$$
By Lemma 5.5.8, there is $\xi < \alpha$ such that for all $\overline{\alpha} \in [\xi, \alpha)$, we know that $(\gamma_0, \ldots, \gamma_{n-1})$ is an initial segment of the walk from β to $\overline{\alpha}$ through $\vec{C}$ and $(\overline{\gamma}_0, \ldots, \overline{\gamma}_{m-1})$ is an initial segment of the walk from $\overline{\beta}$ to $\overline{\alpha}$ through $\vec{C}$.
Choose $\overline{\alpha} \in C_{\gamma_{n-1}} \cap [\xi, \alpha)$. Then $(\gamma_0, \ldots, \gamma_{n-1}, \overline{\alpha})$ is the walk from β to $\overline{\alpha}$ through $\vec{C}$. Since $\rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta})$ for $\overline{\alpha} < \alpha$ by assumption, we know that the walk from $\overline{\beta}$ to $\overline{\alpha}$ through $\vec{C}$ has length $n+1$. This implies that the walk from $\overline{\beta}$ to $\overline{\gamma}_{m-1}$ through $\vec{C}$ has length at most n and we get $m \leq n$.
A symmetric argument shows that $n \leq m$, thus $m = n$.
For all $\overline{\alpha} \in [\xi, \alpha)$, we know that
$$\begin{aligned} &\rho_{\vec{C}}(\gamma_{n-1}, \beta) ^\frown \rho_{\vec{C}}(\overline{\alpha}, \gamma_{n-1}) \\ =&\rho_{\vec{C}}(\overline{\alpha}, \beta) \overset{ass}{=} \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta}) \\ \overset{m=n}{=}&\rho_{\vec{C}}(\overline{\gamma}_{n-1}, \overline{\beta}) ^\frown \rho_{\vec{C}}(\overline{\alpha}, \overline{\gamma}_{n-1}). \end{aligned}$$
This implies that $\rho_{\vec{C}}(\gamma_{n-1}, \beta) = \rho_{\vec{C}}(\overline{\gamma}_{n-1}, \overline{\beta})$, because both sequences have the same length. If $\overline{\alpha} \in [\xi, \alpha)$, then
$$\operatorname{type}\left(C_{\gamma_{n-1}} \cap \overline{\alpha}\right) = \rho_{\vec{C}}(\overline{\alpha}, \beta)(n) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta})(n) = \operatorname{type}\left(C_{\overline{\gamma}_{n-1}} \cap \overline{\alpha}\right).$$
Since $\alpha \in \mathrm{Lim}$, this implies that $\operatorname{type}\left(C_{\gamma_{n-1}} \cap \alpha\right) = \operatorname{type}\left(C_{\overline{\gamma}_{n-1}} \cap \alpha\right)$ and we can

conclude that

$$\begin{aligned}\rho_{\vec{C}}(\alpha,\beta) =&\rho_{\vec{C}}(\gamma_{n-1},\beta)^\frown \rho_{\vec{C}}(\alpha,\gamma_{n-1})\\ =&\rho_{\vec{C}}(\gamma_{n-1},\beta)^\frown \left(\operatorname{type}\left(C_{\gamma_{n-1}}\cap\alpha\right)\right)\\ =&\rho_{\vec{C}}(\overline{\gamma}_{n-1},\overline{\beta})^\frown \left(\operatorname{type}\left(C_{\overline{\gamma}_{n-1}}\cap\alpha\right)\right)\\ =&\rho_{\vec{C}}(\overline{\gamma}_{n-1},\overline{\beta})^\frown \rho_{\vec{C}}(\alpha,\overline{\gamma}_{n-1}) = \rho_{\vec{C}}(\alpha,\overline{\beta}). \quad\square\end{aligned}$$

Proof of Theorem 5.7.9, (2) implies (1). Let κ be an inaccessible cardinal that is Mahlo. Then there is a club set C in κ that consists of singular cardinals. Fix a surjection $\varphi\colon\kappa\to{}^{<\omega}\kappa$ such that $\varphi[\delta]={}^{<\omega}\delta$ for every $\delta\in C$.
There is a C-sequence $\vec{C}=(C_\alpha)_{\alpha<\kappa}$ such that the following first two statements hold for all $\alpha\in\mathrm{Lim}\cap\kappa$ and the last two statements hold for all $\overline{\alpha}\in C_\alpha$:

a. If $\overline{\alpha}=\sup(C\cap\alpha)<\alpha$, then $C_\alpha=(\overline{\alpha},\alpha)$.

b. If $\alpha=\sup(C\cap\alpha)$, then $\operatorname{type}(C_\alpha)=\operatorname{cof}(\alpha)<\min(C_\alpha)$.

c. If $\overline{\alpha}$ is a limit point of C_α, then $\overline{\alpha}\in C$.

d. If $\overline{\alpha}$ is not a limit point of C_α, then $\overline{\alpha}=\gamma+1$ for some $\gamma\in C$.

Claim. *If $\alpha<\beta<\kappa$ with $\alpha\in C$, then $\rho_{\vec{C}}(\alpha,\beta)\in{}^{<\omega}\alpha$.*

Proof. Let $(\gamma_0,\dots,\gamma_n)$ be the walk from β to α through $\vec{C}$ and let $k\le n$ be minimal with $C_{\gamma_k}\cap\alpha$ is unbounded in α.
Then we have $\rho_{\vec{C}}(\alpha,\beta)(i)=\operatorname{type}(C_{\gamma_i}\cap\alpha)<\alpha$ for all $i<k$.
Since $\alpha\in C\cap C_{\gamma_k}$, we know that γ_k is a limit point of C and

$$\operatorname{type}(C_{\gamma_k}\cap\alpha)<\operatorname{type}(C_{\gamma_k})<\min(C_{\gamma_k})<\alpha.$$

This shows that $\rho_{\vec{C}}(\alpha,\beta)(i)<\alpha$ for all $i<n$. $\square$

Since κ is inaccessible, Lemma 5.5.11 implies that $\left|\mathbb{T}(\rho_{\vec{C}})(\alpha)\right|<\kappa$ holds for all $\alpha<\kappa$.
Choose $t=\rho_{\vec{C}}(\cdot,\beta)\restriction\alpha\in\mathbb{T}(\rho_{\vec{C}})$ with $\alpha\in C$. Let $(\gamma_0,\dots,\gamma_n)$ denote the walk from β to α through $\vec{C}$, let $k\le n$ be minimal with $C_{\gamma_k}\cap\alpha$ is unbounded in α and let $\xi<\alpha$ be minimal with $C_{\gamma_i}\cap\alpha\subseteq\xi$ for all $i<k$.
By the above claim, there is $\overline{\alpha}<\alpha$ such that

$$\begin{aligned}&\varphi(\overline{\alpha})=(\operatorname{cof}(\alpha),\xi,k)^\frown\rho_{\vec{C}}(\alpha,\beta)\text{ if }\alpha<\beta\text{ and}\\&\varphi(\overline{\alpha})=(\operatorname{cof}(\alpha),\xi,k)\text{ otherwise.}\end{aligned}$$

Define

$$r(t)=\rho_{\vec{C}}(\cdot,\beta)\restriction\overline{\alpha}\subsetneq t.$$

Claim. *If $\alpha,\overline{\alpha}\in C, s=\rho_{\vec{C}}(\cdot,\overline{\beta})\restriction\overline{\alpha}\in\mathbb{T}(\rho_{\vec{C}})(\overline{\alpha})$ and $t=\rho_{\vec{C}}(\cdot,\beta)\restriction\alpha\in\mathbb{T}(\rho_{\vec{C}})(\alpha)$ with $\overline{\alpha}<\alpha$ and $r(s)=r(t)$, then $s\not\subseteq t$.*

Proof. Assume that $s \subsetneq t$ holds.

1. Assume that $\alpha = \beta$.
   Then $r(s) = r(t)$ implies that $\overline{\alpha} = \overline{\beta}$ and $\operatorname{type}(C_\alpha) = \operatorname{type}(C_{\overline{\alpha}})$.
   By Proposition 5.5.7, $\rho_{\vec{C}}(\cdot, \overline{\alpha}) \subseteq \rho_{\vec{C}}(\cdot, \alpha)$ implies that $C_{\overline{\alpha}} = C_\alpha \cap \overline{\alpha}$, contradiction.

2. Assume that $\alpha < \beta$.
   Let $(\gamma_0, \dots, \gamma_n)$ denote the walk from β to α through $\vec{C}$, $(\overline{\gamma}_0, \dots, \overline{\gamma}_m)$ denote the walk from $\overline{\beta}$ to $\overline{\alpha}$ through $\vec{C}$.
   Let $k \leq n$ be minimal with $C_{\gamma_i} \cap \alpha \subseteq \xi$ for all $i < k$.
   Then $r(s) = r(t)$ implies that $\overline{\alpha} < \overline{\beta}, \rho_{\vec{C}}(\alpha, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta}), m = n, k$ is minimal with $C_{\overline{\gamma}_k} \cap \overline{\alpha}$ unbounded in $\overline{\alpha}$ and $C_{\overline{\gamma}_i} \cap \overline{\alpha} \subseteq \xi$ for all $i < k$.

   2.1. Assume $k = n$. Then
   $$\sup\left(C_{\gamma_{n-1}} \cap \overline{\alpha}\right), \sup\left(C_{\overline{\gamma}_{n-1}} \cap \overline{\alpha}\right) \leq \xi < \overline{\alpha}$$
   and Lemma 5.7.11 implies that $\rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta})$.

   2.2. Assume $k = n-1$ and $\sup\left(C_{\gamma_{n-1}} \cap \overline{\alpha}\right) < \overline{\alpha}$.
   Choose $\delta \in C_{\overline{\gamma}_{n-1}}$ with $\delta > \xi, \sup\left(C_{\gamma_{n-1}} \cap \overline{\alpha}\right)$.
   Then $(\overline{\gamma}_0, \dots, \overline{\gamma}_{n-1}, \delta)$ is the walk from $\overline{\beta}$ to δ through $\vec{C}$, $(\gamma_0, \dots, \gamma_{n-1})$ is an initial segment of the walk from β to δ through $\vec{C}$ and $\min\left(C_{\gamma_{n-1}} \setminus \delta\right) \geq \overline{\alpha} > \delta$.
   Hence,
   $$\operatorname{lh}_{\mathbb{T}(\rho_{\vec{C}})}\left(\rho_{\vec{C}}(\delta, \overline{\beta})\right) = n < \operatorname{lh}_{\mathbb{T}(\rho_{\vec{C}})}\left(\rho_{\vec{C}}(\delta, \beta)\right),$$
   contradiction. Hence $\sup\left(C_{\gamma_{n-1}} \cap \overline{\alpha}\right) = \overline{\alpha}$ and, by Lemma 5.7.11, we get $\rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta})$.

   Using Lemma 5.7.12, we can conclude that
   $$\rho_{\vec{C}}(\alpha, \beta) \neq \rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta}) = \rho_{\vec{C}}(\alpha, \beta),$$
   contradiction.

□

Define
$$\mathbb{S} = \left(\left\{t \in \mathbb{T}(\rho_{\vec{C}}) \mid \operatorname{lh}_{\mathbb{T}(\rho_{\vec{C}})}(t) \in C \cup \{0\}\right\}\right).$$
Then $\mathbb{S}$ is a tree of height κ with $|\mathbb{S}(\alpha)| < \kappa$ for all $\alpha < \kappa$.

Claim. $\mathbb{S}$ *is special.*

Proof. Set $R(\emptyset) = \emptyset$ and
$$R(t) = \rho_{\vec{C}}(\cdot, \beta) \restriction \sup\left\{\alpha \in C \mid \alpha \leq \operatorname{lh}_{\mathbb{T}(\rho_{\vec{C}})}\left(r(t)\right)\right\}$$
for all $t = \rho_{\vec{C}}(\cdot, \beta) \restriction \alpha \in \mathbb{S}$ with $\alpha \in C$.
Then $R \colon \mathbb{S} \to \mathbb{S}$ is regressive.

Choose $t = \rho_{\vec{C}}(\cdot, \beta)\upharpoonright\alpha \in \mathbb{S}$ and let $0 < \lambda < \kappa$ be minimal with $\alpha + \lambda \in C$. Then there is an enumeration $\left(s^\gamma_\xi\right)_{\xi<\delta_\gamma}$ of $\mathbb{T}(\rho_{\vec{C}})(\alpha + \gamma)$ with $\delta_\gamma < \kappa$ for each $\gamma < \lambda$.
Fix $t_0, t_1 \in R^{-1}[\{t\}]$ such that $r(t_0) = r(t_1) = s^\gamma_\xi$ for some $\gamma < \lambda$ and $\xi < \delta_\gamma$. Then t_0 and t_1 are incompatible in $\mathbb{T}(\rho_{\vec{C}})$ and hence incompatible in $\mathbb{S}$. This shows that $R^{-1}[\{t\}]$ is the union of less than κ-many antichains in $\mathbb{S}$. □

This shows that $\mathbb{S}$ is a special κ-Aronszajn tree. □

Theorem 5.7.13 (Aronszajn-Specker). *If κ is an infinite cardinal with $\kappa = \kappa^{<\kappa}$, then there is a special κ^+-Aronszajn tree.* *Lecture 29 2nd Feb*

Proof (Todorcevic). By the assumption, κ is regular and there is a C-sequence $\vec{C} = (C_\alpha)_{\alpha<\kappa^+}$ such that $\mathrm{type}(C_\alpha) = \kappa$ for all $\alpha \in E^{\kappa^+}_\kappa$ (remember that for regular κ, $E^{\kappa^+}_\kappa$ is the set of cardinals $\alpha < \kappa^+$ with $\mathrm{cof}(\alpha) = \kappa$), $\mathrm{type}(C_\alpha) < \kappa$ for all $\alpha \in E^{\kappa^+}_{<\kappa}$ and $\rho_{\vec{C}}(\alpha, \beta) \in {}^{<\omega}\kappa$ for all $\alpha < \beta < \kappa^+$.
By the proof of Specker's Theorem (Theorem 5.5.2), $\mathbb{T}(\rho_{\vec{C}})$ is a κ^+-Aronszajn tree.
Given $t = \rho_{\vec{C}}(\cdot, \beta)\upharpoonright\alpha \in \mathbb{T}(\rho_{\vec{C}})$ with $\alpha \in E^{\kappa^+}_\kappa$, we define

$$c(t) = \begin{cases} \rho_{\vec{C}}(\alpha, \beta), & \text{if } \alpha < \beta. \\ \emptyset, & \text{otherwise.} \end{cases}$$

Claim. *If $\alpha, \overline{\alpha} \in E^{\kappa^+}_\kappa$, $s = \rho_{\vec{C}}(\cdot, \beta)\upharpoonright\overline{\alpha} \in \mathbb{T}(\rho_{\vec{C}})(\overline{\alpha})$ and $t = \rho_{\vec{C}}(\cdot, \beta)\upharpoonright\alpha \in \mathbb{T}(\rho_{\vec{C}})(\alpha)$ with $\overline{\alpha} < \alpha$ and $c(s) = c(t)$, then $s \not\subseteq t$.*

Proof. Assume $s \subseteq t$.

1. Assume $\alpha = \beta$. Then $c(s) = c(t)$ implies $\overline{\alpha} = \overline{\beta}$ and, by Proposition 5.5.7, $\rho_{\vec{C}}(\cdot, \overline{\alpha}) \subseteq \rho_{\vec{C}}(\cdot, \alpha)$ implies $C_{\overline{\alpha}} = C_\alpha \cap \alpha$.
Since $\mathrm{type}(C_\alpha) = \mathrm{type}(C_{\overline{\alpha}}) = \kappa$, this is a contadiction.

2. Assume $\alpha < \beta$. Let $(\gamma_0, \ldots, \gamma_n)$ denote he walk from β to α through $\vec{C}$ and let $(\overline{\gamma}_0, \ldots, \overline{\gamma}_m)$ denote the walk from $\overline{\beta}$ to $\overline{\alpha}$ through $\vec{C}$.
Then $c(s) = c(t)$ implies $\overline{\alpha} < \overline{\beta}, \rho_{\vec{C}}(\alpha, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta})$ and $m = n$. Since we have $\mathrm{type}\left(C_{\gamma_{n-1}} \cap \overline{\alpha}\right) < \kappa = \mathrm{cof}(\overline{\alpha})$, we have

$$\sup\left(C_{\gamma_{n-1}} \cap \overline{\alpha}\right), \sup\left(C_{\overline{\gamma}_{n-1}} \cap \overline{\alpha}\right) < \overline{\alpha}$$

and Lemma 5.7.11 implies $\rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta})$.
Using Lemma 5.7.12, we can conclude that

$$\rho_{\vec{C}}(\alpha, \beta) \neq \rho_{\vec{C}}(\overline{\alpha}, \beta) = \rho_{\vec{C}}(\overline{\alpha}, \overline{\beta}) = \rho_{\vec{C}}(\alpha, \beta),$$

contradiction.

□

Define
$$\mathbb{S} = \Big(\big\{t \in \mathbb{T}(\rho_{\vec{C}}) \mid \mathrm{lh}_{\mathbb{T}(\rho_{\vec{C}})}(t) \in E^{\kappa^+}_{\kappa} \cup \{0\}\big\}, \subseteq \Big).$$
Then $\mathbb{S}$ is a κ^+-Aronszajn tree.
Set $c(\emptyset) = \kappa$, then the resulting function $c\colon \mathbb{S} \to {}^{<\kappa}\kappa \cup \{\kappa\}$ is injective on chains in $\mathbb{S}$. By Theorem 5.7.6, this shows that $\mathbb{S}$ is special. □

Corollary 5.7.14. *If* GCH *holds, then the following statements are equivalent for every uncountable regular cardinal* κ *hat is not the successor of a singular cardinal:*

1. *There is no special* κ*-Aronszajn tree.*
2. κ *is a Mahlo cardinal.*

Remark 5.7.15. The above corollary can be used to give a short proof of a result of Silver which shows that the consistency of

$$\mathsf{ZFC} + \text{“there are no special } \omega_2\text{-Aronszajn trees”}$$

implies the consistency of

$$\mathsf{ZFC} + \text{“there is a Mahlo cardinal”}.$$

A theorem of Mitchell shows that the converse implication also holds.

6 Diamonds

We will now give a brief notion of what will happen in the next semester by introducing the combinatorial principle $\diamondsuit$.

6.1 Diamond Principles and Suslin Trees

Definition 6.1.1 (Jensen)**.** 1. We say that a sequence $(S_\alpha)_{\alpha<\omega_1}$ whit the property $S_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$ is a $\diamondsuit$*-sequence* if for every $A \subseteq \omega_1$, the set

$$\{\alpha < \omega_1 \mid A \cap \alpha = S_\alpha\}$$

is stationary in ω_1.

2. Let $\diamondsuit$ denote the statement that there is a $\diamondsuit$-sequence.

These principles are often called “guessing principles”, because in a certain sense, we guess how subsets of ω_1 look like.

Proposition 6.1.2. $\diamondsuit$ *implies* CH.

Proof. Let $(S_\alpha)_{\alpha<\omega_1}$ be a $\diamondsuit$-sequence. Given $x \subseteq \omega$, the set $\{\alpha < \omega_1 \mid x = S_\alpha\}$ is stationary in ω_1 and therefore non-empty.
Then $i\colon P(\omega) \to \omega_1, x \mapsto \min\{\alpha < \omega_1 \mid x = S_\alpha\}$ is injective. □

Theorem 6.1.3 (Jensen). *$\Diamond$ implies that there is a Suslin tree (i.e. SH is false).*

The following proof is a main example of how to use guessing principles for such results.

Proof. Let $(S_\alpha)_{\alpha<\omega_1}$ be a $\Diamond$-sequence. By induction on $\alpha < \omega_1$, we construct $\mathbb{T}(\alpha) \subseteq {}^\alpha 2$ and injections $i_\alpha \colon \mathbb{T}(\alpha) \to \big[\omega \cdot \alpha, \omega \cdot (\alpha+1)\big)$ such that the following statements hold:

1. $\mathbb{T}(0) = \{\emptyset\}$.
2. If $t \in \mathbb{T}(\alpha)$ for some $\alpha < \omega_1$, then $t \restriction \overline{\alpha} \in \mathbb{T}(\overline{\alpha})$ for all $\overline{\alpha} \leq \alpha$ and $t \cup \{(\alpha, i)\} \in \mathbb{T}(\alpha+1)$ for all $i < 2$.
3. If $\overline{\alpha} \leq \alpha < \omega_1$ and $s \in \mathbb{T}(\overline{\alpha})$, then there is $t \in \mathbb{T}(\alpha)$ with $s \subseteq t$.
4. If $\alpha \in \mathrm{Lim} \cap \omega_1$ and
$$\mathbb{A}_\alpha = \bigcup_{\overline{\alpha}<\alpha} \{t \in \mathbb{T}(\overline{\alpha}) \mid i_{\overline{\alpha}}(t) \in S_\alpha\}$$
is a maximal antichain in $\mathbb{T}_{<\alpha} = \big(\bigcup_{\overline{\alpha}<\alpha} \mathbb{T}(\overline{\alpha}), \subseteq\big)$, then for every $t \in \mathbb{T}(\alpha)$ there is an $s \in \mathbb{A}_\alpha$ with $s \subsetneq t$.

Assume that $\alpha < \omega_1$, and $\mathbb{T}(\overline{\alpha})$ with the above properties are already constructed for all $\overline{\alpha} < \alpha$.
If $\alpha = \overline{\alpha} + 1$, then
$$\mathbb{T}(\alpha) = \big\{t \cup \{(\alpha, i)\} \mid t \in \mathbb{T}(\overline{\alpha}), i < 2\big\}$$
satisfies the above properties. Hence, we may assume that $\alpha \in \mathrm{Lim}$.
Choose $t \in \mathbb{T}_{<\alpha}$ and let $(\alpha_n)_{n<\omega}$ be a strictly increasing cofinal sequence in α with $\alpha_0 = \mathrm{lh}_{\mathbb{T}_{<\alpha}}(t)$.
The above properties impley that there is a sequence $(t_n)_{n<\omega}, t_n \in \mathbb{T}(\alpha_n)$ for all $n < \omega$, with $t_0 = t$ and $t_n \subsetneq t_{n+1}$ for all $n < \omega$. Set
$$t_* = \bigcup_{n<\omega} t_n \in {}^\alpha 2.$$

If $\mathbb{A}_\alpha$ is a maximal antichain in $\mathbb{T}_{<\alpha}$, then we define
$$\mathbb{T}(\alpha) = \{t_* \mid t \in \mathbb{T}_{<\alpha} \text{ with } s \subseteq t \text{ for some } s \in \mathbb{A}_\alpha\}.$$

Otherwise, we define $\mathbb{T}(\alpha) = \{t_* \mid t \in \mathbb{T}_{<\alpha}\}$. In both cases $\mathbb{T}(\alpha)$ is countable and there is an injection $i_\alpha \colon \mathbb{T}(\alpha) \to \big[\omega \cdot \alpha, \omega \cdot (\alpha+1)\big)$.
Choose $t \in \mathbb{T}_{<\alpha}$. If $\mathbb{A}_\alpha$ is a maximal antichain in $\mathbb{T}_{<\alpha}$, then there is some $s \in \mathbb{A}_\alpha$ and $u \in \mathbb{T}_{<\alpha}$ with $s, t \subseteq u \subsetneq u_* \in \mathbb{T}(\alpha)$.
Otherwise, we have $t \subsetneq t_* \in \mathbb{T}(\alpha)$. This shows that $\mathbb{T}(\alpha)$ satisfies the above properties.
Set
$$\mathbb{T} = \bigcup_{\alpha<\omega_1} \mathbb{T}(\alpha) \text{ and } i = \bigcup_{\alpha<\omega_1} i_\alpha \colon \mathbb{T} \to \omega_1.$$
Then $\mathbb{T}$ is a tree of height ω_1.

Claim. *If $\mathbb{A}$ is a maximal antichain in $\mathbb{T}$, then the set*

$$C = \{\alpha < \omega_1 \mid \overbrace{\omega \cdot \alpha = \alpha}^{club} \wedge \mathbb{A} \cap \mathbb{T}_{<\alpha} \text{ is a maximal antichain in } \mathbb{T}_{<\alpha}\}$$

is a club set in ω_1. □

Claim. *Every antichain in $\mathbb{T}$ is countable.*

Proof. Let $\mathbb{A}$ be a maximal antichain in $\mathbb{T}$ and let C be the corresponding club set in ω_1.
Set $A = i[\mathbb{A}] \subseteq \omega_1$.
Then there is some $\alpha \in C$ with $S_\alpha = A \cap \alpha$ and

$$\mathbb{A} \cap \mathbb{T}_{<\alpha} = \{t \in \mathbb{T}_{<\alpha} \mid i(t) \in S_\alpha\} = \mathbb{A}_\alpha$$

is a maximal antichain in $\mathbb{T}_{<\alpha}$.
By construction, every element of $\mathbb{T}(\alpha)$ extends an element of $\mathbb{A} \cap \mathbb{T}_{<\alpha}$.
Since $\mathbb{A}$ is an antichain in $\mathbb{T}$, this implies that $\mathbb{A} \subseteq \mathbb{T}_{<\alpha}$ and $\mathbb{A}$ is countable. □

Claim. *Every chain in $\mathbb{T}$ is countable.*

Proof. Assume that c is a chain in $\mathbb{T}$ with $\mathrm{lh}_{\mathbb{T}}(c) = \omega_1$.
Then

$$f = \bigcup c \colon \omega_1 \to 2 \text{ and } \mathbb{A} = \Big\{ f\restriction\alpha \cup \big\{(\alpha, 1 - f(\alpha))\big\} \mid \alpha < \omega_1 \Big\}$$

is an uncountable antichain in $\mathbb{T}$, contradicting the previous claim. □

This shows that $\mathbb{T}$ is a Suslin tree. □

Definition 6.1.4. Let κ be an uncountable cardinal and $S \subseteq \kappa$. Lecture 30 4th Feb

1. We say that a sequence $(S_\alpha)_{\alpha \in S}$ with $S_\alpha \subseteq \alpha$ for every $\alpha \in S$ is a $\diamondsuit_S$*-sequence* if the set $\{\alpha \in S \mid A \cap \alpha = S_\alpha\}$ is stationary in κ for every $A \subseteq \kappa$.
2. We let $\diamondsuit_S$ denote the statement that there is a $\diamondsuit_S$-sequence.

Remark 6.1.5. Results of Jensen show that it is consistent with the axioms of ZFC that $\diamondsuit_S$ holds for every uncountable regular κ and every stationary subset S of κ.

Proposition 6.1.6. *If κ is an uncountable regular cardinal and $S_0 \subseteq S_1 \subseteq \kappa$, then $\diamondsuit_{S_0}$ implies $\diamondsuit_{S_1}$.* □

Proposition 6.1.7. *If κ is an infinite cardinal, then $\diamondsuit_{\kappa^+}$ implies $2^\kappa = \kappa^+$.* □

Theorem 6.1.8 (Jensen). *If κ is an infinite cardinal that satisfies $\kappa = {}^{<\kappa}\kappa$, then $\diamondsuit_{E^{\kappa^+}_\kappa}$ implies that there is a κ^+-Suslin tree.*

Proof. Let $(S_\alpha)_{\alpha \in E^{\kappa^+}_\kappa}$ be a $\diamondsuit_{E^{\kappa^+}_\kappa}$-sequence. By induction on $\alpha < \kappa^+$, we construct $\mathbb{T}(\alpha) \subseteq {}^\alpha 2$ and injections $i_\alpha \colon \mathbb{T}(\alpha) \to \big[\kappa \cdot \alpha, \kappa \cdot (\alpha + 1)\big)$ such that the following holds:

1. $\mathbb{T}(0) = \{\emptyset\}$.
2. If $t \in \mathbb{T}(\alpha)$ for some $\alpha < \kappa^+$, then $t \cup \{(\alpha, i)\} \in \mathbb{T}(\alpha+1)$ for all $i < 2$ and $t{\restriction}\overline{\alpha} \in \mathbb{T}(\overline{\alpha})$ for all $\overline{\alpha} \leq \alpha$.
3. If $\overline{\alpha} \leq \alpha < \kappa^+$ and $s \in \mathbb{T}(\overline{\alpha})$, then there is $t \in \mathbb{T}(\alpha)$ with $s \subseteq t$.
4. If $\alpha \in E^{\kappa^+}_{<\kappa}$ and $t \in {}^{\alpha}2$ with $t{\restriction}\overline{\alpha} \in \mathbb{T}(\overline{\alpha})$ for all $\overline{\alpha} < \alpha$, then $t \in \mathbb{T}(\alpha)$.
5. If $\alpha \in E^{\kappa^+}_{\kappa}$ and
$$\mathbb{A}_\alpha = \bigcup_{\overline{\alpha}<\alpha} \{t \in \mathbb{T}(\overline{\alpha}) \mid i_{\overline{\alpha}}(t) \in S_\alpha\}$$
is a maximal antichain in $\mathbb{T}_{<\alpha} = \left(\bigcup_{\overline{\alpha}<\alpha} \mathbb{T}(\overline{\alpha}), \subseteq\right)$, then for every $t \in \mathbb{T}(\alpha)$ there is $s \in \mathbb{A}_\alpha$ with $s \subsetneq t$.

Assume that $0 < \alpha < \kappa^+$ and $\mathbb{T}(\overline{\alpha})$ is constructed for all $\overline{\alpha} < \alpha$. If $\alpha = \overline{\alpha} + 1$, then $\mathbb{T}(\alpha)$ with the above properties exists. Hence we may assume $\alpha \in \mathrm{Lim}$.
Fix $t \in \mathbb{T}_{<\alpha}$ and let $(\alpha_\gamma)_{\gamma<\mathrm{cof}(\alpha)}$ be strictly increasing and cofinal in α with $\alpha_0 = \mathrm{lh}_{\mathbb{T}_{<\alpha}}(t)$.
Since $\alpha_\gamma \in E^{\kappa^+}_{<\kappa}$ for every $\gamma \in \mathrm{cof}(\alpha) \cap \mathrm{Lim}$, we can use the above properties to construct $(t_\gamma)_{\gamma<\mathrm{cof}(\alpha)}, t_\gamma \in \mathbb{T}(\alpha_\gamma)$ for all $\gamma < \mathrm{cof}(\alpha)$, with $t_0 = t$ and $t_{\overline{\gamma}} \subsetneq t_\gamma$ for all $\overline{\gamma} < \gamma < \mathrm{cof}(\alpha)$.
Then $t_* = \bigcup_{\gamma<\mathrm{cof}(\alpha)} t_\gamma \in {}^{\alpha}2$.
If $\alpha \in E^{\kappa^+}_{<\kappa}$, then we set $\mathbb{T}(\alpha) = \{t \in {}^{\alpha}2 \mid \forall \overline{\alpha} < \alpha \; t{\restriction}\overline{\alpha} \in \mathbb{T}(\overline{\alpha})\}$.
Then $|\mathbb{T}(\alpha)| \leq \kappa^{<\kappa} = \kappa$ and there is a suitable injection i_α.
Now, assume $\alpha \in E^{\kappa^+}_{\kappa}$. If $\mathbb{A}_\alpha$ is a maximal antichain in $\mathbb{T}_{<\alpha}$, then we define
$$\mathbb{T}(\alpha) = \{t_* \mid t \in \mathbb{T}_{<\alpha} \text{ with } s \subseteq t \text{ for some } s \in \mathbb{A}_\alpha\}.$$
Otherwise, set $\mathbb{T}(\alpha) = \{t_* \mid t \in \mathbb{T}_{<\alpha}\}$. Then $|\mathbb{T}(\alpha)| \leq \kappa$ and i_α exists. Moreover, in all cases the above statements hold. Set
$$\mathbb{T} = \left(\bigcup_{\alpha<\kappa^+} \mathbb{T}(\alpha), \subseteq \right).$$

Claim. *If $\mathbb{A}$ is a maximal antichain in $\mathbb{T}$, then the set*
$$C = \{\alpha < \kappa^+ \mid \kappa \cdot \alpha = \alpha \wedge \mathbb{T} \cap \mathbb{T}_{<\alpha} \textit{ is a maximal antichain in } \mathbb{T}_{<\alpha}\}$$
is a club set in κ^+. □

Claim. *Every antichain in $\mathbb{T}$ is a subset of $\mathbb{T}_{<\alpha}$ for some $\alpha < \kappa^+$.* □

Claim. *Every chain in $\mathbb{T}$ has cardinality at most κ.* □

This shows that $\mathbb{T}$ is a κ^+-Suslin tree. □

6.2 Diamond Principles and the Generalised Continuum Hypothesis

We conclude this lecture by proving a theorem of SHELAH that connects $\Diamond_S$ and GCH.

Theorem 6.2.1 (Shelah)**.** *The following statements are equivalent for every uncountable cardinal κ.*

1. $2^\kappa = \kappa^+$.
2. $\Diamond_S$ *holds for every stationary subset* $S \subseteq \kappa^+$ *of* $\kappa^+ \setminus E^{\kappa^+}_{\mathrm{cof}(\kappa)}$.

Before we proof this we introduce some notation. For a cardinal κ and a set x we define

$$P_\kappa(x) = \{y \in P(x) \mid |y| < \kappa\}.$$

A less often used notation for this is $P_{<\kappa}(x)$.

Proof (Rinot). By Proposition 6.1.7, we only need to show that (1) implies (2).
Let $S \subseteq \kappa^+ \setminus E^{\kappa^+}_{\mathrm{cof}(\kappa)}$ be stationary in κ^+ and assume $2^\kappa = \kappa^+$.

1. Given $\delta < \kappa^+$, let $(A^\alpha_\delta)_{\alpha<\mathrm{cof}(\kappa)}, A^\alpha_\delta \in P_\kappa(\delta\times\delta)$ for all $\alpha < \mathrm{cof}(\kappa)$, be an $\subseteq$-increasing sequence with $\delta \times \delta = \bigcup_{\alpha<\mathrm{cof}(\kappa)} A^\alpha_\delta$.
2. Let $(X_\nu)_{\nu<\kappa^+}$ be an enumeration of $P_{\kappa^+}(\kappa \times \kappa \times \kappa^+)$ which exists by assumption.
3. Given $(\alpha, \tau) \in \kappa \times \kappa$ and $X \subseteq \kappa \times \kappa \times \kappa^+$, set
$$\pi_{\alpha,\tau} = \{\gamma < \kappa^+ \mid (\alpha, \tau, \gamma) \in X\} \subseteq \kappa^+.$$
4. Given $B \subseteq \kappa^+ \times \kappa^+$ and $(\alpha, \tau) \in \kappa \times \kappa$, set
$$(B)_{\alpha,\tau} = \bigcup_{(\mu,\nu)\in B} \pi_{\alpha,\tau}(X_\nu) \subseteq \kappa^+.$$

Assume that for all $(\alpha, \tau) \in \kappa\times\kappa$ there is no sequence $(B_\delta)_{\delta\in S}$ such that $B_\delta \subseteq A^\alpha_\delta \subseteq \delta\times\delta$ for all $\delta \in S$ and $\big((B_\delta)_{\alpha,\tau}\big)_{\delta\in S}$ is not a $\Diamond_S$-sequence.
By induction on $\tau < \kappa$, we inductively construct sequences:

(a) $\big(\{Z^\alpha_\tau \subseteq \kappa^+ \mid \alpha < \kappa\}\big)_{\tau<\kappa}$.

(b) $\big(\{C^\alpha_\tau \subseteq \kappa^+ \mid \alpha < \kappa\}\big)_{\tau<\kappa}$ club sets.

(c) $\big(\{A^\alpha_\delta(\tau) \subseteq A^\alpha_\delta \mid \alpha < \kappa, \delta \in C^\alpha_\tau \cap S\}\big)_{\tau<\kappa}$.

$\tau = 0$: Let $\tau = 0$ and fix $\alpha < \kappa$. Then $\big((A^\alpha_\delta)_{\alpha,0}\cap\delta\big)_{\delta\in S}$ is not a $\Diamond_S$-sequence by assumption. Hence there is

$$Z^\alpha_0 \neq (A^\alpha_\delta)_{\alpha,0} = \bigcup_{(\mu,\nu)\in A^\alpha_\delta} \pi_{\alpha,0}(X_\nu) \cap \delta \text{ for all } \delta \in C^\alpha_0 \cap S.$$

First, assume that there is $(\mu,\nu) \in A^\alpha_\delta$ with $Z^\alpha_0 \cap \mu \neq \pi_{\alpha,0}(X_\nu) \cap \delta$.
Then we define $A^\alpha_\delta(0) = A^\alpha_\delta \setminus \{(\mu,\nu)\}$.
In the other case, if $Z^\alpha_0 \cap \mu = \pi_{\alpha,0}(X_\nu) \cap \delta$ holds for all $(\mu,\nu) \in A^\alpha, \delta$, then $\sup\{\mu < \delta \mid \exists \nu < \kappa^+ \, (\mu,\nu) \in A^\alpha_\delta\} < \delta$, because otherwise

$$Z^\alpha_0 \cap \delta = \bigcup_{(\mu,\nu)\in A^\alpha_\delta} Z^\alpha_0 \cap \mu = \bigcup_{(\mu,\nu)\in A^\alpha_\delta} \pi_{\alpha,0}(X_\nu) \cap \delta,$$

contradiction.
We define $A^\alpha_\delta(0) = A^\alpha_\delta$ in this case.

Up to τ: Assume that the above sequences are constructed up to τ. Set

$$D = \bigcap\{C^\alpha_{\overline{\tau}} \mid \overline{\tau} < \tau, \alpha < \kappa\}.$$

Given $\delta \in D \cap S$ and $\alpha < \kappa$, define

$$B^\alpha_\delta = \bigcap\{A^\alpha_\delta(\overline{\tau}) \mid \overline{\tau} < \tau\} \subseteq A^\alpha_\delta.$$

By the assumption, for every $\alpha < \kappa$, the sequence $\big((B^\alpha_\delta)_{\alpha,\tau} \cap \delta\big)_{\delta\in D\cap S}$ is not a $\diamondsuit_S$-sequence. Given $\alpha < \kappa$, this shows that there is $Z^\alpha_\tau \subseteq \kappa^+$ and a club set C^α_τ in κ^+ with

$$Z^\alpha_\tau \cap \delta \neq (B^\alpha_\delta)_{\alpha,\tau} \cap \delta = \bigcup_{(\mu,\nu)\in B^\alpha_\delta} \pi_{\alpha,\tau}(X_\nu) \cap \delta$$

for all $\delta \in C^\alpha_\tau \cap D \cap S$.
First, assume there is $(\mu,\nu) \in B^\alpha_\delta$ with $Z^\alpha_\tau \cap \mu \neq \pi_{\alpha,\tau}(X_\nu) \cap \delta$.
Then define $A^\alpha_\delta(\tau) = B^\alpha_\delta \setminus \{(\mu,\nu)\}$.
In the other case, if $Z^\alpha_\tau \cap \mu = \pi_{\alpha,\tau}(X_\nu) \cap \delta$ holds for all $(\mu,\nu) \in B^\alpha_\delta$, then $\sup\{\mu < \delta \mid \exists \nu < \kappa^+ \, (\mu,\nu) \in B^\alpha_\delta\} < \delta$ and we set $A^\alpha_\delta(\tau) = B^\alpha_\delta$.

Set

$$Z = \{(\alpha,\tau,\gamma) \mid \alpha < \kappa, \tau < \kappa, \gamma \in Z^\alpha_\tau\}$$

and define

$$f\colon \kappa^+ \to \kappa^+, \quad \mu \mapsto \min\{\nu < \kappa^+ \mid X_\nu = Z \cap (\kappa \times \kappa \times \mu)\}.$$

Since the set of all $\delta < \kappa^+$ with $f[\delta] \subseteq \delta$ is a club in κ^+, we can choose

$$\delta \in S \cap \bigcap_{\tau<\kappa} \bigcap_{\alpha<\kappa} C^\alpha_\tau \text{ with } f[\delta] \subseteq \delta.$$

Then $f{\restriction}\delta \subseteq \delta \times \delta = \bigcup_{\alpha<\operatorname{cof}(\kappa)} A^\alpha_\delta$ and we can define

$$g\colon \delta \to \operatorname{cof}(\kappa), \quad \mu \mapsto \min\{\alpha < \operatorname{cof}(\kappa) \mid (\mu, f(\mu)) \in A^\alpha_\delta\}.$$

Since $\delta = g^{-1}[\mathrm{cof}(\kappa)]$ and $\mathrm{cof}(\delta) \neq \mathrm{cof}(\kappa)$, there is $\alpha_* < \mathrm{cof}(\kappa)$ such that $H = g^{-1}[\alpha_*]$ is cofinal in δ. Since $A_\delta^{\alpha_*} \supseteq \bigcup_{\alpha<\alpha_*} A_\delta^\alpha$, this shows $f{\upharpoonright}H \subseteq A_\delta^{\alpha_0}$.
If $\mu \in H$ with $f(\mu) = \nu$ and $\tau < \kappa$, then

$$Z_\tau^{\alpha_*} \cap \mu = \{\gamma < \mu \mid (\alpha_*, \tau, \gamma) \in Z\} = \pi_{\alpha,\tau}(Z \cap (\kappa \times \kappa \times \mu)) \cap \delta = \pi_{\alpha_*,\tau}(X_\nu) \cap \delta.$$

Since $f{\upharpoonright}H \subseteq A_\delta^{\alpha_0}$ and $Z_\tau^{\alpha_*} \cap \mu = \pi_{\alpha_*,\tau}(X_{f(\mu)}) \cap \delta$ hold for all $\tau < \kappa$, the above definition implies that $f{\upharpoonright}H \subseteq A_\delta^{\alpha_*}(\tau)$ for all $\tau < \kappa$.
In particular, we have

$$\sup\{\mu < \delta \mid \exists \nu < \kappa^+ \, (\mu, \nu) \in A_\delta^{\alpha_*}(\tau)\} \geq \sup(H) = \delta$$

for all $\tau < \kappa$.
By the assumption, we can conclude that $\left(A_\delta^{\alpha_*}(\tau)\right)_{\tau<\kappa}$ is strictly $\subseteq$-decreasing and consists of subsets of $A_\delta^{\alpha_*} \in P_\kappa(\delta \times \delta)$, contradiction.
This shows that there is a pair $(\alpha, \tau) \in \kappa \times \kappa$ and a sequence $(B_\delta)_{\delta \in S}, B_\delta \subseteq A_\delta^\alpha$ for all $\delta \in S$, such that $\left((B_\delta)_{\alpha,\tau} \cap \delta\right)_{\delta \in S}$ is a $\Diamond_S$-sequence. $\square$

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Index

$(<\kappa)^\lambda$, 35
$(\alpha,\beta) \lhd (\alpha',\beta')$, 27
$+_{\mathrm{Ord}}$, 17, 26
$<\mu$-complete, 48
$<_{\mathrm{lex}}$, 17
E^κ_μ, 46
F-positive, 48
F^+, 48
F^*, 48
H_κ, 38
$P_\kappa(x)$, 96
$P_{<\kappa}(x)$, 96
$R(k,k)$, 64
$[\mathbb{T}]$, 55
Card, 22
$\Delta(b,c)$, 56
$\Delta(x,y)$, 80
$\Delta_{x,y}$, 43
$\diamondsuit$, 92
$\diamondsuit$-sequence, 92
$\diamondsuit_S$, 94
$\diamondsuit_S$-sequence, 94
Lim, 46
Ord, 13
$\aleph_0$, 24
$\aleph_\alpha$, 24
α^+, 24
$\bar{\bar{x}}$, 22
$\beta \to (\alpha)^\gamma_\delta$, 61
$\beth$-number, 38
$\beth_\alpha(\kappa)$, 79
$\big(f(i)\big)_{i\in s}$, 30
card(x), 22
$\cdot_{\mathrm{Ord}}$, 17, 26
cof(γ), 32
$\gimel$-function, 36
ht$(\mathbb{T})$, 54
$\in$-comparable, 13
$\kappa^{<\lambda}$, 35
$\leq_{\mathrm{ub}}$, 52
$\mathrm{lh}_{\mathbb{T}}(c)$, 55
$\mathrm{lh}_{\mathbb{T}}(t)$, 54
$\lim_{\alpha<\beta}\gamma_\alpha$, 26
$\mathbb{N}$, 19
$\mathbb{N}^+$, 19
$\mathbb{Q}$, 19
$\mathbb{R}$, 19
$\mathbb{T}(\alpha)$, 54
$\mathbb{T}(\rho_{\vec{C}})$, 72
$\mathbb{T}_s$, 54
$\mathbb{T}_{<\alpha}$, 54
$\mathcal{B}$, 30
$\mathcal{C}(\mathbb{R},\mathbb{R})$, 28
$\mathcal{C}_\kappa$, 48
$\mathcal{F}$, 45
ω, 12, 13
ω_α, 24
$\partial\mathbb{T}$, 55
$\pi_{(A,<)}$, 16
$\preceq_{\mathrm{lex}}$, 68
$\mathrm{pred}_<(x)$, 11
$\prod_{i\in s} X_i$, 31
$\prod_{i\in s} \kappa_i$, 31
rank$[x]$, 39
$\rho_{\vec{C}}(\alpha,\beta)$, 72
$\rho_{\vec{C}}(\cdot,\beta)$, 72
root$(\mathbb{T})$, 54
σ-algebra, 29
$\sigma\mathbb{T}$, 55
$\mathrm{succ}_{\mathbb{T}}(s)$, 54
$\sum_{i\in s} \kappa_i$, 30
$\sup x$, 13
tc$(\{x\})$, 15
type$(x,<_0)$, 23
$|x|$, 22
${}^AB$, 8
${}^{<\alpha}X$, 28
$x \preceq y$, 21
${}^{<\lambda}x$, 35
CH, 35
GCH, 35
SH, 66, 67

ZF, 18
ZFC, 25
ZF^-, 10

almost disjoint
  function, 51
  set of functions, 51
Aronszajn tree, 59, 65
Aronszajn's Theorem, 71
Axiom of Choice, 25

Bernstein-Cantor theorem, 21
Borel σ-algebra
  on $\mathbb{R}$, 29
Borel codes, 29
Borel measurable, 29
Borel set, 29
branch, 55
  cofinal, 55
  maximal, 55
Burali-Forti paradox, 14

C-sequence, 71
  $\square(\kappa)$-sequence, 76
  $\square_\kappa$-sequence, 78
  coherent, 76
Cantor's paradox, 24
Cantor's Theorem, 22
cardinal, 22
  inaccessible, 38
  limit, 24
  Mahlo, 83
  regular, 34
  singular, 34
  successor, 24
  tree property, 59
  weakly compact, 81
chain, 41
  downwards-closed, 55
  maximal element, 41
  strict upper bound, 41
  upper bound, 41
choice function, 25
  partial, 42
class, 5
  proper, 5
  transitive, 13
class term, 5
closure point
  of a function, 46
club filter, 48
cofinal, 32
cofinality, 32
collapsing map, 16
colouring, 61
composition, 7
continuum function, 36
continuum hypothesis, 35
  generalised, 35

dense, 44, 60
diagonal intersection, 47
domain, 7

Easton's Theorem, 36, 53
equivalence class, 8
equivalence relation, 8
Erdős-Rado Theorem, 79

filter, 45
  club, 48
  dual ideal of a filter, 48
  Frechet, 45
  positive sets, 48
  principal, 45
  ultrafilter, 45
flip set, 43
Fodor's Lemma, 48
formal language
  set theory, 3
formula
  $\in$-formula, 3
  set theoretic, 3
Frechet filter, 45
full code, 72
function, 3, 7
  almost disjoint, 51
  Borel measurable, 29
  closure point, 46
  cofinal, 32

continuous, 26
sequentially, 44
standard topology of ℝ?, 44
with respect to a topology, 44
from A to B, 8
injective, 8
injective on chains, 84
Lebesgue measurable, 30
left inverse, 40
one-to-one, 8
onto, 8
partial, 8
regressive, 48
on a tree, 84
strictly increasing, 26
strictly monotone, 26
surjective, 8
weakly increasing, 26
weakly monotone, 26
fundamental property of ordered pairs, 6

Gödel ordering, 27
Gödel pairing, 27

Hausdorff Maximality Principle, 56
Hausdorff's formula, 37

ideal, 45
dual ideal of a filter, 48
image, 7
induction
classes, 15
natural numbers, 14
sets, 11
intersection
diagonal, 47

König's Lemma, 59
König's Theorem, 31
Kurepa's Theorem, 57, 67

Lebesgue measurable, 30
limit point, 46

map, 7
monotone enumeration, 16
Mostowski collapse, 16
Mostowski's isomorphism theorem, 16

number
natural, 3
rational, 3
real, 3

order
chain, 41
lexicographical, 17
linear, 8
strict, 8
maximal element, 41
partial, 8
strict, 8
tree, 54
strict upper bound, 41
upper bound, 41
order type, 23
ordinal, 13
limit ordinal, 13
successor ordinal, 13

pair
ordered, 3, 6
perfect kernel, 13
preimage, 7

Ramification Lemma, 62
Ramsey number, 64
Ramsey's Theorem, 64
Finite, 64
range, 7
rank, 18
recursion
classes, 15
sets, 11
relation, 7
antisymmetric, 8
connex, 8
equivalence, 8
extensional, 16
irreflexive, 7

linear, 8
reflexive, 7
symmetric, 7
transitive, 8
well-founded, 11
well-order, 11
restriction, 7
Russell's paradox, 4

σ-algebra, 29
sequence, 30
$\diamondsuit$-sequence, 92
$\diamondsuit_S$-sequence, 94
C-sequence, 71
set
almost disjoint, 51
Borel, 29
bounded, 46
closed, 46
in κ, 46
sequentially, 44
standard topology of $\mathbb{R}$, 44
club, 46
co-finite, 45
cofinal, 32
compatible in a tree $\mathbb{T}$, 54
countable, 22
direct successor in a tree, 54
empty, 4, 5
finite, 22
flip, 43
hereditary size $< \kappa$, 38
homogeneous for a colouring, 61
inductive, 9
infinite, 22
open
standard topology of $\mathbb{R}$, 44
stationary, 46
unbounded, 32, 46
uncountable, 23
Silver's Theorem, 53
Solovay's Theorem, 49
Specker's Theorem, 71
Suslin Hypothesis, 66, 67
Suslin line, 67
Suslin tree, 67

topological space
connected, 65
countable chain condition (c.c.c.), 66
irreducible, 60
noetherian, 60
separable, 66
topology
base, 65
connected, 65
dense subset, 66
order topology, 65
separable, 66
standard topology of $\mathbb{R}$, 44
transitive closure, 15
transitive collapse, 16
transversal, 42
tree, 54
antichain, 67
Aronszajn tree, 59, 65
κ-Aronszajn tree, 59
ω_1-Aronszajn tree, 65
branch, 55
cofinal, 55
maximal, 55
chain, 55
length, 55
compatible, 54
direct successor, 54
extensional at limit levels, 57
length of t, 54
level, 54
regressive function, 84
special, 84
suitable linear order, 68
$\preceq_{\text{lex}}$, 68
Suslin tree, 67
κ-Suslin tree, 67
ω_1-Suslin tree, 67
well-founded labeled, 29
tree property, 59
triple

ordered, 3
tuple, 6

Ulam's Theorem, 48
ultrafilter, 45
unbounded, 32
universe of sets, 5
upper bound, 41
strict, 41

Von Neumann hierarchy, 18

walk, 72
full code, 72
well-founded labeled tree, 29
well-order, 11
Well-ordering Theorem, 25

z-approximation, 12
ZF-Axioms
Extensionality, 5
Foundation, 9
Infinity, 9
Pairing, 6
Power Set, 18
Replacement Scheme, 10
Separation Scheme, 9
Set Existence, 4
Union, 8
Zorn's Lemma, 41