3. Boolean-valued models

3.1. **Background.** For constructing a model of ZFC, the first problem is that one cannot prove in our base theory ZFC that there exist set models of ZFC by the second incompleteness theorem. There is no place to explain the proof of incompleteness here, but we sketch the following weaker variant:

Remark 3.1. ZFC cannot prove the existence of a transitive set model $(M, \in)$ of ZFC. To see this, let α be least such that there is such a model $(N, \in)$ with $\mathrm{Ord}^N = \alpha$. Let $(N, \in)$ be such a model. Now suppose $(N, \in) \models$ "there exists a transitive set model $(M, \in)$ of ZFC". One can show that the statement "$(M, \in) \models \mathsf{ZFC}$" is absolute between $(N, \in)$ and $(V, \in)$, since N is transitive and satisfies enough recursion to define the satisfaction relation $\models$.

Since there might not exist a set model of ZFC, one can instead try to construct a proper class model. It is natural to try to construct a transitive proper class model, but the problem is that such models do not necessarily exist:

Remark 3.2. Suppose that $V = L$, where L is the constructible universe (see e.g. [Jech: Set theory, chapter 13]). This is a minimal transitive proper class model of ZFC, i.e. if $V = L$, then there do not exist any transitive class models $M \subsetneq V$. Then there do not exist any wellfounded proper class models either, since the Mostowski collapse would map such a model to an isomorphic transitive model.

It is not clear how one would directly construct an illfounded model. However, there is a natural strategy that works:

Construct a class model with truth values in a Boolean algebra.

In fact, this will also give rise to an illfounded proper class (actual) model of set theory. Note that an alternative (and more common) approach is to start with a transitive set model $M \in V$ of a fragment ZFC^* of ZFC and construct a model $M[G]$ of ZFC^* in V with $M \subseteq M[G]$. This model $M[G]$ will come up later as well. The two approaches are equivalent.

We first want to motivate truth values in a Boolean algebra. Think of a situation where you are not sure whether a property holds. For instance, take a random variable $\xi\colon \mathbb{R} \to \mathbb{R}$ and the property "$\xi > 0$". Since ξ might take positive and negative values, we want to give this formula a "fuzzy" value, for example the set $\{x \in \mathbb{R} \mid \xi(x) > 0\}$, or instead its equivalence class up to Lebesgue null sets.

In general, one needs to have operations $\wedge$, $\vee$ and $\neg$ on truth values that correspond to the operations on formulas. In other words, the truth values should form a Boolean algebra. We will thus study Boolean algebras. The Lebesgue meaesurable sets of reals modulo null sets are an example of a Boolean algebra.

I would like to point out a difference to *continuous logic*, where all truth values in the interval $[0, 1]$. Regarding the example of random variables, one could try to define the truth value of "$\xi > 0$" for a random variable $\xi\colon [0, 1] \to \mathbb{R}$ as the measure of the set $\{x \in \mathbb{R} \mid \xi(x) > 0\}$. But then one cannot tell from the truth values of two statements φ and ψ alone whether they are compatible: given that φ is true with probability 60% and ψ with probability 30%, we cannot tell whether both φ and ψ can hold at the same time. Ini contrast, the truth values of incompatible statements φ and ψ are incompatible elements a, b of a Boolean algebra, i.e. $a \wedge b = 0$.

3.2. **Boolean algebras.** A *partial order* $\leq$ is a reflexive, transitive and antisymmetric.[6] binary relation on a set B. For example, the truth values 0 (false) and 1 (true) form a partial order $B = \{0, 1\}$ with $0 \leq 1$. B has natural operations $\vee = \max$ and $\wedge = \min$. B is the most basic example of a *Boolean algebra*.

Definition 3.3. A *Boolean algebra* is a partially ordered set $(B, \leq)$ with a least element 0, a largest element 1 and the properties:

(1) Any $x, y \in B$ have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

[6]Recall that symmetric means that $\forall x\ x \leq x$, transitive that $\forall x, y, z\ x \leq y \wedge y \leq z \to x \leq z$ and antisymmetric that $\forall x, y\ x \leq y \wedge y \leq x \to x = y$.

(2) $\vee$ and $\wedge$ are distribute over each other, i.e.

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

(3) Every $x \in B$ has a complement, i.e., an element $-x$ such that

$$x \vee -x = 1$$

$$x \wedge -x = 0$$

$(B, \leq)$ is called *complete* if for every subset X of B, a greatest lower bound $\inf(X)$ and a least upper bound $\sup(X)$ exist.[7]

Conversely, one can also define $\leq$ from $\wedge$, since $x \leq y$ holds if and only if $x \wedge y = x$.

Lemma 3.4. *$-x$ is unique.*

Proof. Suppose that y, z are complements of x, i.e. both satisfy the properties of $-x$. Using $x \vee y = 1$ and $x \wedge z = 0$, we have $z = (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) = y \wedge z$, so $z \leq y$. Similarly $y \leq z$. Thus $y = z$. □

Example 3.5.

(1) The Boolean algebra $\{0, a, -a, 1\}$ with $0 < a, -a < 1$.
(2) The power set $\mathscr{P}(S)$ of a set S partially ordered by $\subseteq$ with $\vee = \cup$, $\wedge = \cap$, and $-$ is complementation.

Suppose that $\mathbb{B}$ is a subalgebra of $\mathscr{P}(S)$ for some set S. Then every $x \in S$, all $a \in \mathbb{B}$ with $x \in a$, are compatible, i.e. have nonempty intersection. In other words, they generate a filter (see below). Stone's representation theorem reconstructs S from $\mathbb{B}$ via filters on $\mathbb{B}$.

Definition 3.6. Suppose that $\mathbb{B}$ is a Boolean algebra.

(1) A *filter* on $\mathbb{B}$ is a subset F of $\mathbb{B}$ with
    (a) $1 \in F$ and $0 \notin F$.
    (b) If $a, b \in F$, then $a \wedge b \in F$.
    (c) If $a \in F$ and $a \leq b$, then $b \in F$.
(2) An *ultrafilter* on $\mathbb{B}$ is a filter U on $\mathbb{B}$ such that for every $a \in \mathbb{B}$: $a \in U$ or $-a \in U$.

It is easy to see that ultrafilters are precisely the maximal filters.

For every Boolean algebra $\mathbb{B}$ and any $a \in \mathbb{B}$, there exists an ultrafilter U on $\mathbb{B}$ with $a \in \mathbb{B}$. Such an ultrafilter can be constructed by induction by going through an enumeration of all elements of $\mathbb{B}$. In the successor step, one has a filter F on $\mathbb{B}$ and $a \in \mathbb{B}$. One shows that $\forall\, b \in F\ a \wedge b \neq 0$ or $\forall\, b \in F\ (-a) \wedge b \neq 0$ holds. Otherwise there are $b, b' \in F$ with $a \wedge b = (-a) \wedge b' = 0$. Then

$$0 = (a \wedge (b \wedge b')) \vee (-a \wedge (b \wedge b')) = b \wedge b' \in F.$$

But $0 \notin F$, since F is a filter.

Theorem 3.7 (Stone's representation theorem)**.** *Every Boolean algebra $\mathbb{B}$ is isomorphic to a subalgebra of $(\mathscr{P}(S), \subseteq)$ for some set S.*

Proof sketch. The *Stone space* $S(\mathbb{B})$ of $\mathbb{B}$ is defined as the set of ultrafilters on $\mathbb{B}$. The basic open sets are:

$$N_a := \{U \in S(\mathbb{B}) \mid a \in U\}$$

for $a \in \mathbb{B}$. $S(\mathbb{B})$ is a zero-dimensional Hausdorff space. Moreover, it is compact, since it is homeomorphic to a subspace of the product $\prod_{a \in \mathbb{B}} \mathbf{2}$ by sending an ultrafilter to its characteristic function. Here $\mathbf{2}$ denotes the discrete space with two elements. It is easy to check that the image is closed. $\prod_{a \in \mathbb{B}} \mathbf{2}$ is compact by Tykhonov's theorem. □

[7] It suffices that $\sup(X)$ exists for every subset X.

3.3. **Boolean-valued models.** We first return to the previous example. Consider all Lebesgue measurable subsets of $\mathbb{R}$ modulo null sets. I.e. let $A \leq B$ if $A \setminus B$ is null and $A \sim B$ if $A \leq B \leq A$. Let $[A]$ denote the $\sim$-equivalence class of a Lebesgue measurable set A. Let $\mathbb{B}$ denote the Boolean algebra of equivalence classes ordered by $\leq$. Now consider the set of random variables in $\mathbb{R}$, i.e. Lebesgue measurable functions $\xi\colon \mathbb{R} \to \mathbb{R}$, as the underlying set of our structure. For any Lebesgue measurable subset A of $\mathbb{R}$, let $[\![\xi \in A]\!] := \xi^{-1}(A)$ be the truth value of the statement "$\xi \in A$".

How much structure of the reals can be carried over to this set? For example, note that the random variables are not linearly ordered. Let ξ_+, ξ_- be characteristic functions or $\mathbb{R}_+$ and $\mathbb{R}_-$. None is pointwise large than the other. Similarly, the set of random variables does not form a field with the pointwise operations $+$ and $\cdot$.

However, if we define the Boolean value

$$[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!],$$

then

$$[\![(\xi_+ \leq \xi_-) \vee (\xi_- \leq \xi_+)]\!] \sim \mathbb{R} = 1_{\mathbb{B}}.$$

So the set of random variables as a $\mathbb{B}$-valued model (see the next definition) is in fact linearly ordered by $\leq$. One can similarly check that it is a field.

Problem 3.8. Show that the equivalence classes of Lebesgue measurable subsets of $\mathbb{R}$ equipped with the partial order $A \leq B$ if $A \setminus B$ is null form a complete Boolean algebra.

Definition 3.9 (Boolean-valued models)**.** Suppose that $\mathbb{B}$ is a complete Boolean algebra and $\mathcal{L}$ is a first-order language. A $\mathbb{B}$-valued model $\mathcal{M}$ in the language $\mathcal{L}$ consists of an underlying set M, whose elements are called *names*, and an assignment of Boolean values $[\![s = t]\!]$, $[\![R(s_0, ..., s_n)]\!]$ and $[\![y = f(s_0, ..., s_n)]\!]$ in $\mathbb{B}$ to atomic formulas with parameters $s, t, s_0, ..., s_n \in M$. These assignments must follow the laws of equality:

$$[\![s = s]\!] = 1$$

$$[\![s = t]\!] = [\![t = s]\!]$$

$$[\![s = t]\!] \wedge [\![t = u]\!] \leq [\![s = u]\!]$$

$$\bigwedge_{i<n} [\![s_i = t_i]\!] \wedge [\![R(\vec{s})]\!] \leq [\![R(\vec{t})]\!]$$

If the language includes functions symbols, then additionally:

$$\bigwedge_{i<n} [\![s_i = t_i]\!] \wedge [\![y = f(\vec{s})]\!] \leq [\![y = f(\vec{t})]\!]$$

$$\bigvee_{t \in M} [\![t = f(\vec{s})]\!] = 1.$$

$$[\![t_0 = f(\vec{s})]\!] \wedge [\![t_1 = f(\vec{s})]\!] \leq [\![t_0 = t_1]\!].$$

The requirements for functions assert that the equality axiom holds for functions and any function takes a unique value.

One can then extend the Boolean values to all formulas by recursion on formulas:

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$$

$$[\![\neg\varphi]\!] = \neg[\![\varphi]\!]$$

$$[\![\exists x \varphi(x, \vec{s})]\!] = \bigvee_{t \in M} [\![\varphi(t, \vec{s})]\!]$$

It follows that $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$. Moreover, one can check by induction on φ that the general equality axiom now has Boolean value 1:

$$[\![\vec{s} = \vec{t} \wedge \varphi(\vec{s}) \to \varphi(\vec{t}))]\!] = 1.$$

3.4. **Turning Boolean-valued models into actual models.** Suppose that $\mathcal{M}$ is a Boolean-valued model in a language $\mathcal{L}$ and T is an $\mathcal{L}$-theory, i.e. a set of $\mathcal{L}$-sentences. By definition, $\mathcal{M}$ is a model of T, denoted, $\mathcal{M} \models T$, if $[\![\varphi]\!] = 1$ for all $\varphi \in T$.

A Boolean-valued model M can be turned into an actual model under an additional assumption. M is called *full* if for every $\mathcal{L}$-formula $\varphi(x, \vec{y})$ and $\vec{s} \in M^n$, there is some $t \in M$ such that $[\![\exists x \varphi(x, \vec{s})]\!] = [\![\varphi(t, \vec{s})]\!]$.

Proposition 3.10. *If a theory T has a full $\mathbb{B}$-valued model $\mathcal{M}$, then it has a model.*

Proof. Let U be an ultrafilter on $\mathbb{B}$. We define a model $\mathcal{M}/U$ by letting

$$\mathcal{M}/U \models \varphi(\sigma_0, \ldots, \sigma_n) \iff [\![\varphi(\sigma_0, \ldots, \sigma_n)]\!] \in U$$

for an atomic formulas $\varphi(x_0, \ldots, x_n)$ and $\sigma_0, \ldots, \sigma_n \in \mathcal{M}$. The same equivalence then holds for all formulas by induction. The interesting case is

$$\begin{aligned} \mathcal{M}/U \models \exists x \varphi(x, \vec{\sigma}) &\iff \exists x \in M \ \mathcal{M}/U \models \varphi(x, \vec{\sigma}) \\ &\iff \exists x \in M \ [\![\varphi(x, \vec{\sigma})]\!] \in U \\ &\iff [\![\exists x \varphi(x, \vec{\sigma})]\!] \in U \end{aligned}$$

The first equivalence holds by definition of satisfaction, the second by the induction hypothesis and the last one holds by fullness. □

Corollary 3.11. *The following are equivalent for any full $\mathbb{B}$-valued model:*
(a) $a \leq [\![\varphi(\sigma)]\!]$
(b) $\mathcal{M}/U \models \varphi(\sigma)$ *for every ultrafilter U on $\mathbb{B}$ with $a \in U$.*

Proof. If $a \leq [\![\varphi(\sigma)]\!]$ and $a \in U$, then $[\![\varphi(\sigma)]\!] \in U$, so $\mathcal{M}/U \models \varphi(\sigma)$. If $a \not\leq [\![\varphi(\sigma)]\!]$, then $b := a \cap [\![\neg\varphi(\sigma)]\!] \neq 0$. Find an ultrafilter U on $\mathbb{B}$ with $b \in U$. Then $\mathcal{M}/U \models \neg\varphi(\sigma)$. □

Example 3.12 (Ultrapowers). Suppose that I is a set and $\mathcal{M}_i = (M_i, R_i)$ is an $\mathcal{L}$-structure for each $i \in I$. Let $\mathbb{B} = \mathscr{P}(I)$. The underlying set consists of all functions $f \in M := \prod_{i \in I} M_i$ and the Boolean values are defined as

$$[\![\varphi(f_0, ..., f_n)]\!] = \{i \in I \mid M_i \models \varphi(f_0(i), ..., f_n(i))\}$$

for atomic formulas φ. Los' theorem states that this equality holds for all formulas. This is a special case of Remark 3.10, assuming this $\mathbb{B}$-valued model is full. Why is it full? Let $J := [\![\exists x \varphi(x, \vec{\sigma})]\!] = \bigcup_{f \in M} [\![\varphi(f, \vec{\sigma})]\!]$. Then for each $j \in J$, there exists some $f(j) \in M_j$ with $\mathcal{M}_j \models \varphi(f(j), \vec{\sigma})$. Let $f(i) \in M_i$ be arbitrary for $i \in I \setminus J$. Then $[\![\varphi(f, \vec{\sigma})]\!] = J$.

3.5. **Boolean-valued models of set theory.** We describe how to define a $\mathbb{B}$-valued class model of set theory for any complete Boolean algebra $\mathbb{B}$. The class of $\mathbb{B}$-names, denoted $V^{\mathbb{B}}$, is defined by recursion: τ is a $\mathbb{B}$-name if τ is a set of pairs (σ, b), where σ is a $\mathbb{B}$-name and $b \in \mathbb{B}$. The atomic Boolean values are defined by recursion with the use of an auxiliary relation symbol $\subseteq$.

$$[\![\tau \in \sigma]\!] = \bigvee_{(\eta, b) \in \sigma} [\![\tau = \eta]\!] \wedge b$$

$$[\![\tau = \sigma]\!] = [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!]$$

$$[\![\tau \subseteq \sigma]\!] = \bigwedge_{\eta \in \mathrm{dom}(\tau)} ([\![\eta \in \tau]\!] \rightarrow [\![\eta \in \sigma]\!])^8$$

One then extends the Boolean value assignment by recursion to all formulas $\varphi(\sigma_0, \ldots, \sigma_n)$ in the language of set theory as above for $\mathbb{B}$-valued models. We still need to verify the axioms of equality for Boolean-valued models:

Lemma 3.13. *For all $s, t, u \in V^{\mathbb{B}}$:*
(1) $[\![s = s]\!] = 1$
(2) $[\![s = t]\!] = [\![t = s]\!]$.
(3) $[\![s = t]\!] \wedge [\![t = u]\!] \leq [\![s = u]\!]$.

[8] It is easy to see that this equals $\bigwedge_\xi ([\![\xi \in \tau]\!] \rightarrow [\![\xi \in \sigma]\!])$. For each name η, $[\![\xi \in \tau]\!] = \bigvee_{\langle \eta, b \rangle \in \tau} [\![\eta = \xi]\!] \wedge b$ and $b \leq [\![\eta \in \tau]\!]$. Since $[\![\tau \subseteq \sigma]\!]$ is stronger than $[\![\eta \in \tau]\!] \rightarrow [\![\eta \in \sigma]\!]$, it is thus stronger than $[\![\xi \in \tau]\!] \rightarrow [\![\xi \in \sigma]\!]$.

(4) (a) $[\![s = t]\!] \wedge [\![u \in s]\!] \leq [\![u = t]\!]$.
(b) $[\![s = t]\!] \wedge [\![s \in u]\!] \leq [\![t = u]\!]$.

Proof. See handwritten proof in the lecture notes in the "notes" folder on teams. $\square$

3.6. ZFC **in** $V^{\mathbb{B}}$**.** Fix a complete Boolean algebra $\mathbb{B}$. We shall first show that $V^{\mathbb{B}}$ is a $\mathbb{B}$-valued model of ZF. The choice axiom follows later.[9] It suffices to show that the extension, union, power set, infinity and foundation axioms and each instance of the replacement scheme hold with Boolean value 1.

The **extension** axiom is true in $V^{\mathbb{B}}$ by definition of $[\![\sigma \in \tau]\!]$.

For the **union** axiom, take $\sigma \in V^{\mathbb{B}}$. We show that

$$\tau := \{\langle \xi, a \wedge b\rangle \mid \exists \langle \eta, a\rangle \in \sigma : \ \langle \xi, b\rangle \in \eta\}$$

is a name for $\bigcup \sigma$, i.e. $[\![\forall x (x \in \tau \leftrightarrow \exists y \in \sigma\ x \in y)]\!] = 1$. Recall that by definition, $[\![\forall x (x \in \tau \leftrightarrow \exists y \in \sigma\ x \in y)]\!] = \bigwedge_{\zeta \in V^{\mathbb{B}}} [\![\zeta \in \tau]\!] \leftrightarrow [\![\exists x \in \sigma\ \zeta \in x]\!]$. We fix $\zeta \in V^{\mathbb{B}}$ and show $[\![\zeta \in \tau]\!] = [\![\exists x \in \sigma\ \zeta \in x]\!]$. Recall that

$$[\![\zeta \in \tau]\!] = \bigvee_{(\xi,b)\in\tau} [\![\zeta = \xi]\!] \wedge b = \bigvee_{(\eta,a)\in\sigma} \bigvee_{(\xi,b)\in\eta} [\![\zeta = \xi]\!] \wedge a \wedge b$$

$$[\![\exists x \in \sigma\ \zeta \in x]\!] = \bigvee_{\rho} ([\![\rho \in \sigma]\!] \wedge [\![\zeta \in \rho]\!]) = \bigvee_{\rho} \bigvee_{(\eta,a)\in\sigma} \bigvee_{(\xi,b)\in\rho} [\![\eta = \rho]\!] \wedge a \wedge [\![\zeta = \xi]\!] \wedge b$$

We obtain $\leq$ by letting $\rho = \eta$, and $\geq$ since for any $(\xi, b) \in \rho$, $[\![\eta = \rho]\!] \wedge [\![\zeta = \xi]\!] \wedge b \leq [\![\zeta \in \eta]\!] \leq \bigvee_{(\xi,c)\in\eta} [\![\zeta = \xi]\!] \wedge c$.

For the **power set** axiom, suppose that τ is a $\mathbb{B}$-name. We show that

$$\sigma = \{\langle \eta, b\rangle \mid \eta \subseteq \mathrm{dom}(\tau) \times \mathbb{B} \wedge b = [\![\eta \subseteq \tau]\!]\}$$

is a name for the power set of τ. We want to show $[\![\forall x\ (x \in \sigma \leftrightarrow x \subseteq \tau)]\!] = 1$, i.e. for all names ν, $[\![\nu \in \sigma]\!] = [\![\nu \subseteq \tau]\!]$. We have $[\![\nu \in \sigma]\!] = \bigvee_{\langle \eta,b\rangle\in\sigma} [\![\nu = \eta]\!] \wedge b = \bigvee_{\langle \eta,b\rangle\in\sigma} [\![\nu = \eta \wedge \eta \subseteq \tau]\!] \leq [\![\nu \subseteq \tau]\!]$. Conversely, consider the name

$$\nu' := \{\langle \eta, b\rangle \mid \eta \in \mathrm{dom}(\tau) \wedge b = [\![\eta \in \nu \cap \tau]\!]\}$$

for any name ν. It is easy to check that ν' is a name for $\nu \cap \tau$, i.e. $[\![\nu' = \nu \cap \tau]\!] = 1$.[10] Hence $[\![\nu \subseteq \tau \to \nu = \nu']\!] = 1$. Since $\nu' \subseteq \mathrm{dom}(\tau) \times \mathbb{B}$, it follows that $\nu' \in \mathrm{dom}(\sigma)$. We further have $[\![\nu' \in \sigma]\!] \geq [\![\nu' \subseteq \tau]\!]$ by the definition of σ.[11] Thus $[\![\nu \leq \tau]\!] \leq [\![\nu \in \sigma]\!]$.

For the **infinity** axiom, suppose that ω is an inductive set (i.e. it is closed under $+1$) and let $\sigma = \check{\omega}$. We claim that σ is a name for an inductive set. Since σ is a check-name, we have

$$[\![\forall x \in \sigma\ (x + 1 \in \sigma)]\!] = \bigwedge_{n\in\omega} [\![\check{n} + 1 \in \sigma]\!],$$

so it suffices that $[\![\check{n} + 1 \in \sigma]\!] = \bigvee_{m\in\omega} [\![\check{m} \in \check{n} + 1]\!] \geq [\![(n+1)\check{} = \check{n} + 1]\!]$ equals 1. Since $(n \cup \{n\})\check{} = \check{n} \cup \{\langle \check{n}, 1\rangle\}$ by the definition of check-names, one can now verify $[\![\xi \in (n+1)\check{}\ ]\!] = [\![\xi \in \check{n} + 1]\!]$ as required.

For the **foundation** axiom, take a name σ. Let $a = [\![\sigma \neq \emptyset]\!]$. Suppose that $b = [\![\forall x \in \sigma\ \exists y \in x \cap \sigma]\!] \wedge a > 0$. Let η be of least rank with $[\![\eta \in \sigma]\!] \wedge b > 0$. We claim that $b \leq [\![\forall x \in \eta\ x \notin \sigma]\!]$. It suffices that for any ξ,

$$[\![\xi \in \eta]\!] \wedge [\![\xi \in \sigma]\!] \wedge b = 0.$$

Since

$$[\![\xi \in \eta]\!] = \bigvee_{\langle \epsilon,c\rangle\in\eta} [\![\xi = \epsilon]\!] \wedge c,$$

we have $[\![\xi = \epsilon]\!] \wedge [\![\xi \in \sigma]\!] \wedge b \leq [\![\epsilon \in \sigma]\!] \wedge b = 0$ as required.

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[9] While this could be proved directly, it is immediate from lemmas below.

[10] $[\![\nu' \subseteq \nu \cap \tau]\!] = 1$ follows from $[\![\eta \in \nu']\!] = \bigvee_{\langle \xi,b\rangle\in\nu'} [\![\eta = \xi]\!] \wedge b = \bigvee_{\xi\in\mathrm{dom}(\nu')} [\![\eta = \xi]\!] \wedge [\![\xi \in \nu \cap \tau]\!] \leq [\![\eta \in \nu \cap \tau]\!]$. $[\![\nu \cap \tau \subseteq \nu']\!]$ follows from $[\![\eta \in \nu \cap \tau]\!] = \bigvee_{\langle \zeta,b\rangle\in\tau} [\![\eta = \zeta]\!] \wedge b \wedge [\![\eta \in \nu]\!] \leq \bigvee_{\xi\in\mathrm{dom}(\nu')} [\![\eta = \xi]\!] \wedge [\![\xi \in \nu \cap \tau]\!]$, since $b \leq [\![\zeta \in \tau]\!]$.

[11] $[\![\nu' \in \sigma]\!] = \bigvee_{\langle \eta,b\rangle\in\sigma} [\![\eta = \nu']\!] \wedge b \leq [\![\nu' \subseteq \nu]\!]$, since $b = [\![\eta \subseteq \tau]\!]$.

The **replacement** scheme is shown by separation and collection. (Replacement is equivalent to their conjunction.) For the separation scheme, suppose that τ is a name and $\varphi(x)$ is a formula. We claim that

$$\rho := \{\langle \sigma, b\rangle \mid \sigma \in \mathrm{dom}(\tau),\ b = [\![\sigma \in \tau \wedge \varphi(\sigma)]\!]\}$$

is a name for the subset of τ defined by φ. For any name η, $[\![\eta \in \rho]\!] \leq [\![\eta \in \tau]\!] \wedge [\![\varphi(\eta)]\!]$ by the definition of ρ. For the converse, recall that

$$[\![\eta \in \rho]\!] = \bigvee_{\zeta \in \mathrm{dom}(\tau)} [\![\eta = \zeta]\!] \wedge [\![\zeta \in \tau \wedge \varphi(\zeta)]\!]$$

$$[\![\eta \in \tau]\!] = \bigvee_{\langle \xi, a\rangle \in \tau} [\![\eta = \xi]\!] \wedge a$$

by definition of ρ. For each $\langle \xi, a\rangle \in \tau$,

$$([\![\eta = \xi]\!] \wedge a) \wedge [\![\varphi(\eta)]\!] \leq [\![\eta = \xi]\!] \wedge [\![\xi \in \tau \wedge \varphi(\xi)]\!]$$

as required.

The collection scheme states that for any set a and any formula $\varphi(x, y)$ such that $\forall x \in a\ \exists y\ \varphi(x, y)$ holds, there exists some b such that $\forall x \in a\ \exists y \in b\ \varphi(x, y)$. Fix a name σ. We can let

$$\tau := \{\langle \nu, 1\rangle \mid \nu \in V_\alpha \cap V^{\mathbb{B}}\}$$

for a sufficiently large ordinal α, so that all possible Boolean values realized by witnesses ξ with $\varphi(\sigma, \xi)$ are realized by witnesses in $V_\alpha \cap V^{\mathbb{B}}$, using collection in V. In more detail, by definition

$$[\![\forall x \in a\ \exists y \in b\ \varphi(x, y)]\!] = \bigwedge_\eta [\![\eta \in \sigma]\!] \to \bigvee_\xi [\![\xi \in \tau]\!] \wedge [\![\varphi(\eta, \xi)]\!].$$

One can see as in footnote 8 that $\bigwedge_\eta$ can be equivalently replaced by $\bigwedge_{\eta \in \mathrm{dom}(\sigma)}$. Moreover, $[\![\forall x \in a\ \exists y\ \varphi(x, y)]\!]$ is the same but with $[\![\xi \in \tau]\!]$ omitted. So it suffices to choose τ, α as above so that for any $\eta \in \mathrm{dom}(\sigma)$ and any ξ with $[\![\varphi(\eta, \xi)]\!] > 0$, there is some $\zeta \in V_\alpha$ with $[\![\varphi(\eta, \zeta)]\!] = [\![\varphi(\eta, \xi)]\!]$. Then $[\![\zeta \in \tau]\!] = 1$ follows. Since there are only $|\mathbb{B}|$ many Boolean values, such an α exists by collection.

3.7. $V^{\mathbb{B}}$ **is full.** We want to check that $V^{\mathbb{B}}$ is full so that $V^{\mathbb{B}}/U$ is a model of ZFC for any ultrafilter U on $\mathbb{B}$.

We use the following terminology for partial orders. A subset A of a partial order $\mathbb{P}$ is called a *chain* if $p \leq q$ or $q \leq p$ for all $p, q \in A$. We say that $p, q \in \mathbb{P}$ are *compatible* ($p \parallel q$) if there exists some $r \leq p, q$ and *incompatible* ($p \perp q$) otherwise. A is called an *antichain* if any $p \neq q$ in A are incompatible. An antichain is *maximal* below b if $\forall p \in A\ p \leq b$ and there is no antichain with this property that properly extends A. It is maximal if it is maximal below 1.

Moreover, we call elements of a partial order *conditions*; this reflects the fact that $a \leq [\![\varphi]\!]$ says that a is a sufficient condition for φ to hold, i.e. if U is an ultrafilter on $\mathbb{B}$ with $a \in U$, then $V^{\mathbb{B}}/U \models \varphi$.

Recall that we call the elements of $V^{\mathbb{B}}$ names. The next lemma shows how to combine several names into one.

Lemma 3.14 (Mixing lemma)**.** *Suppose that $A \subseteq \mathbb{B}$ is an antichain and $\langle \tau_a \mid a \in A\rangle$ is a sequence of names. Then there exists a name τ such that $a \leq [\![\tau = \tau_a]\!]$ for all $a \in A$.*

Proof. Let

$$\tau = \{\langle \sigma, b \wedge a\rangle \mid \langle \sigma, b\rangle \in \tau_a \wedge a \in A\}.$$

The point is that this name looks exactly like τ_a if we care only about conditions below a. (We will see below that there is a more semantic way of deriving all kinds of equations like this; one interprets σ^U recursively by only considering conditions in an ultrafilter U.)

Fix $a \in A$. We must verify $a \leq [\![\tau \subseteq \tau_a]\!]$ and $a \leq [\![\tau_a \subseteq \tau]\!]$. Recall

$$[\![\tau \subseteq \tau_a]\!] = \bigwedge_{\eta \in \mathrm{dom}(\tau)} ([\![\eta \in \tau]\!] \rightarrow [\![\eta \in \tau_a]\!]),$$

so we want $a \wedge [\![\eta \in \tau]\!] \leq [\![\eta \in \tau_a]\!]$. By definitions,

$$[\![\eta \in \tau]\!] = \bigvee_{\langle \sigma, b\rangle \in \tau} [\![\sigma = \eta]\!] \wedge b = \bigvee_{a' \in A} \bigvee_{\langle \sigma, b\rangle \in \tau_{a'}} [\![\sigma = \eta]\!] \wedge b \wedge a'$$

$$[\![\eta \in \tau_a]\!] = \bigvee_{\langle \sigma, b\rangle \in \tau_a} [\![\sigma = \eta]\!] \wedge b.$$

Since A is an antichain, $a \wedge [\![\eta \in \tau]\!] = \bigvee_{\langle \sigma, b\rangle \in \tau_a} [\![\sigma = \eta]\!] \wedge b \wedge a \leq [\![\eta \in \tau_a]\!]$.

Moreover, $a \leq [\![\tau_a \subseteq \tau]\!]$ holds since $a \wedge [\![\eta \in \tau_a]\!] \leq [\![\eta \in \tau]\!]$ is immediate from the definitions. □

Lemma 3.15 (Fullness principle)**.** *$V^{\mathbb{B}}$ is full.*

Proof. Consider $\exists x\ \varphi(x, \vec{\tau})$, where $\vec{\tau}$ is a finite sequence of names.

By definition, the Boolean value $b = [\![\exists x \varphi(x, \vec{\tau})]\!]$ is the join of the set $S = \{[\![\varphi(\sigma, \vec{\tau})]\!] \mid \sigma \in V^{\mathbb{B}}\}$. Let D be the downwards closure of S. It is dense below b by the definition of b. Let A be a maximal antichain below b with $A \subseteq D$. Then $\bigvee A = b$. For each $a \in A$, choose some σ_a with $a \leq [\![\varphi(\sigma_a, \vec{\tau})]\!]$. By the mixing lemma, find a name σ such that $a \leq [\![\sigma = \sigma_a]\!]$ for all $a \in A$. Then $a \leq [\![\varphi(\sigma, \vec{\tau})]\!]$ for each $a \in A$ by the equality axioms. So $b = \bigvee A \leq [\![\varphi(\sigma, \vec{\tau})]\!]$.

Conversely, $[\![\varphi(\sigma, \vec{\tau})]\!] \leq \bigvee S = b$ since $[\![\varphi(\sigma, \vec{\tau})]\!] \in S$. □

3.8. **Partial orders and their completions.** This material can be found e.g. in [Jech: Set theory, Theorem 7.13].

In this section, we show that any partial order $\mathbb{P}$ satisfying a mild condition (separativity) is dense in a complete Boolean algebra $\mathbb{B}$. It is easy to see that the notion of $\mathbb{P}$-generic filter and $\mathbb{B}$-generic filter are equivalent. So in fact, the two approaches are in fact equivalent.

Partial orders are used instead of Boolean algebras in actual forcing constructions and to prove properties of forcings such as chain conditions and closure.

3.9. **Large continuum.** We have seen that we can assume $M \subseteq N$ are a class models of ZFC, M is a transitive subclass of N, G is $\mathbb{P}$-generic over M for some $\mathbb{P} \in M$ and $N = M[G] = \{\mathrm{val}(\sigma, G) \mid \sigma \in M \text{ is a } \mathbb{P}\text{-name}\}$, i.e. N is a $\mathbb{P}$-generic extension of M. (Actually, we have seen this for $\mathbb{B}(\mathbb{P})$-names, but for every $\mathbb{B}(\mathbb{P})$-name σ there is a $\mathbb{P}$-name τ with $[\![\sigma = \tau]\!] = 1$.) We will also write σ^G for $\mathrm{val}(\sigma, G)$.

We will show that $\neg\mathsf{CH}$ is consistent. To this end, we want a generic extension $M[G]$ such that $M[G]$ has at least ω_2 new reals and ω_2 is preserved as a cardinal. Work in M. Let $\mathrm{Fun}(X, Y)$ denote the set of finite partial functions from X to Y, ordered by *reverse inclusion*, i.e. $p \leq q := p \supseteq q$. Fix an infinite cardinal κ and let

$$\mathbb{P} := \mathrm{Fun}(\kappa \times \omega, 2).$$

For each $\alpha < \kappa$, let

$$\sigma_\alpha := \{\langle \check{n}, p\rangle \mid p(\alpha, n) = 1\}.$$

Suppose that G is a $\mathbb{P}$-generic filter over M.

Lemma 3.16. *For any $\alpha < \beta < \kappa$, $M[G] \models \sigma_\alpha^G \neq \sigma_\beta^G$.*

Proof. Suppose that $b := [\![\sigma_\alpha = \sigma_\beta]\!] > 0$ in $\mathbb{B}(\mathbb{P})$. Since $\mathbb{P}$ is dense in $\mathbb{B}(\mathbb{P})$, there is some $p \leq b$ in $\mathbb{P}$. Since p is a finite partial function, there exists some $q \leq p$ in $\mathbb{P}$ with

$$q(\alpha, n) \neq q(\beta, n)$$

for some $n \in \omega$. Assume $q(\alpha, n) = 1$ and $q(\beta, n) = 0$. By definition of $\mathrm{val}(\sigma_\alpha, G)$, we have in $M[G]$ that $n \in \sigma_\alpha^G$, but $n \notin \sigma_\beta^G$. □

Thus, if we choose $\kappa = \omega_2$ and both ω_1 and ω_2 remain cardinals in $M[G]$, then $M[G] \models \neg\mathsf{CH}$. The function mapping $\alpha < \omega_2$ to σ_α^G is injective. Cardinal preservation will follow from the *countable chain condition* (c.c.c.). A forcing $\mathbb{P}$ has the c.c.c. if it has no uncountable antichains. (Equivalently, $\mathbb{B}(\mathbb{P})$ has no uncountable chains.) To show the c.c.c., we need a combinatorial lemma:

Lemma 3.17. *(Δ-system lemma) Suppose that $\mathcal{S}$ is an uncountable set consisting of finite sets. Then there exists an uncountable family $\mathcal{S}_0 \subseteq \mathcal{S}$ and a finite set r_0 such that for all distinct $s, t \in \mathcal{S}_0$, $s \cap t = r_0$. $\mathcal{S}_0$ is called a* Δ-system *with root r_0.*

Proof. We can assume that $\mathcal{S}$ has size ω_1 and all elements of $\mathcal{S}$ have the same size n. Let $\langle s_\alpha \mid \alpha < \omega_1\rangle$ enumerate $\mathcal{S}$. For each $i < n$, define $f_i \colon \omega_1 \to \omega_1$ by letting $f_i(\alpha)$ be the ith element of s_α.

Let $k < n$ be least such that $\mathrm{ran}(f_k)$ is unbounded in ω_1. Such a k has to exists, since otherwise $\bigcup_{\alpha<\omega_1} s_\alpha$ is countable and thus $\mathcal{S}$ would be countable. Note that for $k < j < n$, $\mathrm{ran}(f_j)$ is also unbounded.

There exists some $\gamma < \omega_1$ such that $\bigcup_{i<k} \mathrm{ran}(f_i) \subseteq \gamma$. Since there are only countably many finite subsets of α, there exists a finite subset r_0 of α such that for uncountably many $\alpha < \omega_1$, $\{f_i(\alpha) \mid i < k\} = r_0$. Let I denote the set of α with this property.

Since $\mathrm{ran}(f_k)$ is unbounded in ω_1, we can construct by recursion a strictly uncreasing sequence $\langle \alpha_\beta \mid \beta < \omega_1\rangle$ in I such that $f_k(\alpha_0) > \max(r_0)$ and for all $\beta < \omega_1$, $f_k(\alpha_\beta) > \max(s_{\beta'})$ for all $\beta' < \beta$. Then $\mathcal{S}_0 := \{s_{\alpha_\beta} \mid \beta < \omega_1\}$ is a Δ-system with root r_0. □

Lemma 3.18. *Suppose that $\mathbb{P}$ is a c.c.c. forcing in M and $M[G]$ is a $\mathbb{P}$-generic extension of M. Then M and $M[G]$ have the same cardinals.*

Proof. Towards a contradiction, suppose that $\kappa < \lambda$ are infinite cardinals in M and a $\mathbb{P}$-name $\dot{f}$ such that $b := [\![\dot{f} \colon \kappa \to \lambda \text{ is a surjective function}]\!] > 0$.

For each $\alpha < \kappa$ and $\beta < \lambda$, let $b_{\alpha,\beta} := [\![\dot{f}(\alpha) = \beta]\!] \wedge b$. Then for each $\alpha < \kappa$,

$$A_\alpha := \{b_{\alpha,\beta} > 0 \mid \beta < \lambda\}$$

is an antichain, so it is countable by the c.c.c. Since κ, λ are cardinals and $\kappa < \lambda$, there exists some $\beta < \lambda$ such that $b_{\alpha,\beta} = 0$ for all $\alpha < \kappa$.

Suppose that H is a $\mathbb{P}$-generic filter over M with $b \in H$. Then in $M[H]$, $\dot{f}^H(\alpha) \neq \beta$ for all $\alpha < \kappa$. So $\dot{f}^H$ is not surjective onto λ. This contradicts the fact that $M[H] \cong M^{\mathbb{B}(\mathbb{P})}/H$ and $b \in H$. □

The next lemma is an application of the Δ-system lemma.

Lemma 3.19. *For any cardinal κ,* $\mathrm{Fun}(\kappa, 2)$ *has the c.c.c.*

Proof. Towards a contradiction, suppose that A is an uncountable antichain in $\mathbb{P}$. Let $\mathcal{S} := \{\mathrm{dom}(f) \mid f \in A\}$. $\mathcal{S}$ is an uncountable set of finite subsets of κ. By Lemma 3.17, there exists an uncountable Δ-system $\mathcal{S}_0 \subseteq \mathcal{S}$ with root r. For each $s \in \mathcal{S}_0$, pick some $p_s \in A$ with $\mathrm{dom}(p_s) = s$. Let $A_0 := \{p_s \mid s \in \mathcal{S}_0\}$.

Since A_0 is infinite, there exist some $p \neq q$ in A_0 with $p \restriction r = q \restriction r$. Then p and q are compatible, as $\mathrm{dom}(p) \cap \mathrm{dom}(q) = r$. $\square$

Note that $\mathrm{Fun}(\kappa \times \omega, 2)$ is isomorphic to the *finite support product* $\prod_{i \in \kappa} \mathrm{Fun}(\omega, 2)$, where

$$\prod_{i \in I} \mathbb{P}_i$$

is equipped with the coordinatewise partial order, where each $\mathbb{P}_i$ is a partial order.

One can show that a generic filter for this product is of the form $\prod_{i \in I} G_i$, where each G_i is $\mathbb{P}_i$-generic over M.

3.10. **The continuum hypothesis.** Gödel showed in 1940 that the continuum hypothesis is consistent with the axioms of ZFC. To show this, he defined the *constructible universe* L. L is the minimal transitive class model of ZFC that contains all ordinals, i.e. $L \subseteq M$ for all such models M. Gödel showed that L satisfies the continuum hypothesis CH.

One can also easily find a forcing $\mathbb{P}$ that forces CH, i.e. $[\![\mathsf{CH}]\!]_{\mathbb{B}(\mathbb{P})} = 1$.

3.11. **The axiom of choice in $V^{\mathbb{B}}$.** We omitted the axiom of choice above and only showed ZF in $V^{\mathbb{B}}$. To show that AC holds in $V^{\mathbb{B}}$, it suffices to show that it holds in $M[G]$, where G is $\mathbb{P}$-generic over M for some forcing $\mathbb{P}$. ($V^{\mathbb{B}}$ believes that it is of this form.)

It suffices that in $M[G]$, every set σ^G is wellordered. It suffices that σ^G is an image of a wellordered set. But σ^G is a subset of $\{\tau^G \mid \tau \in \mathrm{dom}(\sigma)\}$ and this an image σ. Moreover, σ can be wellordered since $\sigma \in M$ and M is a model of ZFC, so the wellordering theorm holds in M.

3.12. **General remarks about forcing.**

Remark 3.20. If G is $\mathbb{P}$-generic over M, then $M[G]$ is the least model N of ZFC with $M \subseteq N$ and $G \in N$. I.e. $M[G] \subseteq N$ for any such model N. To see this, note that any such model N can compute $\mathrm{val}(\sigma, G)$ for any $\mathbb{P}$-name $\sigma \in M$. Thus $M[G] \subseteq N$.

Recall that a *forcing* is by definition a partial order (i.e. a partially ordered set). Sometimes one means more generally a quasiorder, i.e. a relation that satisfies the conditions on partial orders except antisymmetry: $x \leq y \wedge y \leq x \rightarrow x = y$.

Recall that a forcing $\mathbb{P}$ is called *separative* if $\forall p, q\ p \not\leq q \rightarrow \exists r \leq p : r \perp q$. If $\mathbb{P}$ is separative, then $\mathbb{P}$ is isomorphic to a dense subset of $\mathbb{B}(\mathbb{P})$.

Remark 3.21. Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are separative forcings. We say that $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent* if $\mathbb{B}(\mathbb{P}) \cong \mathbb{B}(\mathbb{Q})$, where $\cong$ denotes isomorphism. In particular, $\mathbb{P}$ and $\mathbb{B}(\mathbb{P})$ are equivalent. More generally, this holds if $\mathbb{P}$ is a dense subset of $\mathbb{Q}$: one can check that for any regular open subset A of $\mathbb{P}$, its upwards closure $\{q \in \mathbb{Q} \mid \exists p \in A\ p \leq q\}$ is a regular open subset of $\mathbb{Q}$ and for any regular open subset B of $\mathbb{Q}$, $B \cap \mathbb{P}$ is a regular open subset of $\mathbb{P}$.

Working in a model M of ZFC, suppose that $\mathbb{P}$ is a dense subset of $\mathbb{Q}$. If G is a $\mathbb{P}$-generic filter over M, then the upwards closure $H := \{q \in \mathbb{Q} \mid \exists p \leq q\ p \in G\}$ is $\mathbb{Q}$-generic over M as well. Conversely, if H is $\mathbb{Q}$-generic over M, then $G := H \cap \mathbb{P}$ is $\mathbb{P}$-generic over M. Thus $\mathbb{P}$ and $\mathbb{Q}$ give rise to the same generic extensions. If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent this holds as well, since both give rise to the same extensions as $\mathbb{B}(\mathbb{P})$.

The *forcing relation* $\Vdash$ is defined by $p \Vdash \varphi(\sigma_0, \ldots, \sigma_n)$ if $p \leq [\![\varphi(\sigma)]\!]$, where $p \in \mathbb{P}$, σ is a $\mathbb{P}$-name and $\varphi(x_0, \ldots, x_n)$ is a formula. Note that the forcing relation for $\mathbb{P}$ talks only about $\mathbb{P}$, not about the Boolean completion.

Remark 3.22. We do not need the Boolean-valued model $V^{\mathbb{B}(\mathbb{P})}$ to prove facts about forcing with $\mathbb{P}$. Instead, assume that M is a transitive class model of ZFC with $\mathbb{P} \in M$ such that for every $p \in \mathbb{P}$, there exists a $\mathbb{P}$-generic filter over M. We know from above that there exists such a model M that is elementarily equivalent to V.

Then $p \Vdash \varphi(\sigma) \iff p \leq [\![\varphi(\sigma)]\!] \iff M^{\mathbb{B}(\mathbb{P})}/G \models \varphi(\sigma)$ for any $\mathbb{B}(\mathbb{P})$-generic filter G over M $\iff M[G] \models \varphi(\sigma)$ for any $\mathbb{P}$-generic filter G over M. This follows from Corollary 3.11 and the fact that $M^{\mathbb{B}(\mathbb{P})}/G \cong M[G]$ if G is a $\mathbb{P}$-generic filter over M. The equivalence

$$p \Vdash \varphi(\sigma) \iff M[G] \models \varphi(\sigma) \text{ for any } \mathbb{P}\text{-generic filter } G \text{ over } M$$

is known as the *forcing theorem.*

Remark 3.23. A remark about the presentation. One can alternatively present forcing by first defining $M[G]$ and proving ZFC in $M[G]$ and the forcing theorem for M as in Kunen's book. Then one has to argue why one can assume that M is countable, in order to know that generic filters over M exist. Note that there might not exist any countable models of ZFC.

3.13. **Cohen forcing.** We want to understand Cohen and random forcing in more detail. *Cohen forcing* $\mathbb{C}$ is defined as $\mathrm{Fun}(\omega, 2)$, the set of finite partial functions $p\colon \omega \to 2$, ordered by inclusion. Instead, we can work with the dense subforcing consisting of all $p\colon n \to 2$ for any $n \in \omega$.

Suppose that M is a model of ZFC and G is Cohen generic over M. Let

$$x_G = \bigcup G = \bigcup_{p \in G} p.$$

Since G is a filter, for any $p, q \in G$, we have $p \cup q \in G$. Thus $x_G = \bigcup G\colon \omega \to 2$ is a partial function.

We claim that $x_G\colon \omega \to 2$ is a total function. To see this, let

$$D_n := \{p \in \mathbb{P} \mid n \in \mathrm{dom}(p)\}$$

for each $n \in \omega$. Since D_n is dense, $G \cap D_n \neq \emptyset$ and hence $\mathrm{dom}(x_G) = \omega$.

x_G is called a Cohen real over M.

A *real* is by definition a function $f\colon \omega \to 2$ or a subset of ω.

We claim that one can reconstruct G from x_G. Let $x = x_G$ and

$$G_x := \{p \in \mathbb{P} \mid p \subseteq x\}.$$

We claim that $G = G_x$. The inclusion $\subseteq$ is obvious. To see that $\supseteq$ holds, suppose that $p \in G_x$, i.e. $p \subseteq x$. For each $n \in \mathrm{dom}(p)$, find $p_n \in G$ with $n \in \mathrm{dom}(p_n)$. Then $q := \bigcup_{n \in \mathrm{dom}(p)} p_n \in G$, since G is a filter. Since $p \subseteq q$, we have $p \in G$.

It follows that the Cohen real x_G generates $M[G]$ in the sense that $M[G]$ is the least transitive model N of ZFC with $M \subseteq N$ and $x_G \in N$.

3.14. **Characterising Cohen reals.** As before, suppose that M is a transitive model of ZFC. We want to characterise Cohen reals over M. On the way, we also determine the Boolean completion $\mathbb{B}(\mathbb{C})$.

The Cantor space 2^ω is equipped with the product topology. The basic open sets are

$$N_t = \{x \in 2^\omega \mid t \subseteq x\}$$

for $t \in 2^{<\omega}$.

A subset A of 2^ω is called *nowhere dense* if for every open subset U of 2^ω, there is some open subset W of U that is disjoint from A. A subset A of 2^ω is called *meager* if $A = \bigcup_{n \in \omega} A_n$, where each A_n is nowhere dense. (I.e. the meager sets form the σ-ideal generated by the nowhere dense sets.) A is called *comeager* if its complement if meager.

If A is nowhere dense, then its closure is also nowhere dense. Therefore, any comeager set contains a subset of the form $\bigcup_{n \in \omega} U_n$, where each U_n is open dense.

The Borel sets form the smallest collection of subsets of 2^ω that contains all open sets and is closed under forming complements and countable unions.

Lemma 3.24. *(Baire category theorem) Any nonempty open set is non-meager.*

Proof. Suppose that $U \neq \emptyset$ is an open subset of 2^ω. Towards a contradiction, suppose that $\langle U_n \mid n \in \omega \rangle$ is a sequence of open dense sets with $U \cap \bigcap_{n\in\omega} U_n = \emptyset$.

We construct a strictly increasing sequence $\langle t_n \mid n \in \omega \rangle$ in $2^{<\omega}$ as follows. Pick some t with $N_t \subseteq U$. Since U_n is open and dense, pick some $t_0 \supseteq t$ with $N_{t_0} \subseteq U_0$.

Given t_n, pick some $t_{n+1} \supseteq t$ with $N_{t_{n+1}} \subseteq U_{n+1}$.

Then $x := \bigcup_{n\in\omega} t_n \in U \cap \bigcap_{n\in\omega} U_n$, but this set is empty. □

A subset A of 2^ω has the *property of Baire* if there exists some open set U such that

$$A \triangle U := (A \setminus U) \cup (U \setminus A)$$

is meager. In this case, one also writes $A =_* U$.

Lemma 3.25. *The collection of all subsets of 2^ω with the property of Baire forms a σ-algebra, i.e. it contains $\emptyset$ and 2^ω and is closed under countable unions and complements.*

Proof. For countable unions, suppose that $A = \bigcup_{n\in\omega} A_n$ and for each $n \in \omega$, U_n is open with $A_n =_* U_n$. It is easy to see that $A =_* \bigcup_{n\in\omega} U_n$.

For complements, suppose that U is open with $A =_* U$. Let $W = 2^\omega \setminus \mathrm{cl}(U)$, where $\mathrm{cl}(U)$ denotes the closure of U. Then $U \cup W$ is open dense, so its complement is nowhere dense and thus meager. Hence $2^\omega \setminus A =_* 2^\omega \setminus U =_* W$. □

Since all open sets have the property of Baire by definition, it follows that every Borel set has the property of Baire.

Let $\mathbb{Q}$ denote the set of all Borel subsets of 2^ω. For $A, B \in \mathbb{Q}$, let $A \leq B$ if $A \setminus B$ is meager. Let $A \sim B$ if $A \leq B$ and $B \leq A$.

Exercise 3.26. *$\mathbb{Q}/\!\sim$ is a complete Boolean algebra.*

It is easy to see that $\mathbb{Q}/\!\sim$ is a Boolean algebra. For suprema, take a countable subset $X = \{[A_n] \mid n \in \omega\}$ of $\mathbb{Q}$. One can show that $\sup_{A\in X}[A] = [\bigcup_{n\in\omega} A_n]$. For an uncountable subset X of $\mathbb{Q}$, one a sequence $\langle A_n \mid n \in \omega \rangle$ with $[A_n] \in X$ such that for all A with $[A] \in X$ and all $n \in \omega$, we have that $A \setminus \bigcup_{n\in\omega} A_n$ is meager. Then $\sup(X) = [\bigcup_{n\in\omega} A_n]$.

Let $\mathbb{Q}_0$ denote the set of all non-meager Borel subsets of 2^ω.

We claim that $\mathbb{C} = \mathrm{Fun}(\omega, 2) = 2^{<\omega}$ is isomorphic to a dense subset of $\mathbb{Q}_0/\!\sim$. To see this, we identify $t \in \mathbb{C}$ with the $\sim$-equivalence class $[N_t]$ in $\mathbb{Q}_0/\!\sim$. The order on $\mathbb{C}$ and its image in $\mathbb{Q}_0/\!\sim$ is the same by the Baire category theorem.

To see that $\mathbb{C}$ (with this identification) is dense in $\mathbb{Q}_0/\!\sim$, take any $A \in \mathbb{Q}_0$. Since A has the property of Baire, there exists an open set U with $A =_* U$. Since A is non-meager, U is nonempty. Find some $t \in 2^{<\omega}$ with $N_t \subseteq U$. Then $N_t \leq A$.

To characterise Cohen reals, we work with *Borel codes.* A Borel code for a Borel set B is some $a \in 2^\omega$ that codes how A is built up from basic open sets N_t. One writes B_a for the Borel set coded by a.

To define Borel codes, fix a bijective function $f\colon \omega \to \omega \times \omega$. For any $x \in 2^\omega$ and $n \in \omega$, let $x_n \in 2^\omega$ denote the function with $x_n(i) = x(n,i)$ for all $i \in \omega$.

Fix an enumeration $\vec{t} = \langle t_i \mid i \in \omega \rangle$ of $2^{<\omega}$. Suppose that $a \in 2^\omega$. We write $a \equiv n$ if a is a constant sequence with value n.

- a codes N_{t_n} if $a_0 \equiv 0$ and $a_1 = \langle 0, \dots, 0, 1, 1, \dots \rangle = 0^n{}^\frown 1^\omega$.
- a codes $2^\omega \setminus B_b$ if $a_0 \equiv 1$ and $a_1 = b$.
- a codes $\bigcup_{n \in \omega} B_{a_n}$ if $a_0 \equiv 2$ and $a_{n+1} = b_n$ for all $n \in \omega$.

An element of 2^ω is called a *Borel code* if it is generated by the above steps.

One can now check (see e.g. [Jech: Set Theory, Lemma 26.4):

Lemma 3.27. *The following are equivalent for a real x:*

(a) *x is a Cohen real over M.*
(b) *$x \in B_a$ for every Borel code $a \in M$ such that B_a is comeager.*

This involves a bit of work, for example one needs to show that for a Borel code a and a real x, the statement "$x \in B_a$" is absolute between transitive models M, N of ZFC that contain both a and x. This can be shown by induction on the Borel rank.

It follows immediately from the definition the following condition characterises Cohen reals x: $x \in \bigcup_{i \in \omega} N_{t_i}$ for every sequence $\vec{t} = \langle t_i \mid i \in \omega \rangle$ such that $\bigcup_{i \in \omega} N_{t_i}$ is an (open) dense subset of 2^ω.

3.15. **Random forcing.** Again, we work with the Cantor space 2^ω. (One could also work with the real line.)

Let μ denote the uniform measure on 2^ω. I.e. μ is the completion of the unique measure μ on the Borel subsets of 2^ω with $\mu(N_t) = 2^{-|t|}$ for all $t \in 2^{<\omega}$.

Let $\mathbb{Q}$ denote the set of all Borel subsets of 2^ω. For $A, B \in \mathbb{Q}$, let $A \leq B$ if $A \setminus B$ is null (i.e. has measure 0). Let $A \sim B$ if $A \leq B$ and $B \leq A$.

Exercise 3.28. $\mathbb{Q}/{\sim}$ *is a complete Boolean algebra.*

(This was done in one of the exercises.) For a countable subset $X = \{[A_n] \mid n \in \omega\}$ of $\mathbb{P}$, $\sup_{A \in X}[A] = [\bigcup_{n\in\omega} A_n]$. For an uncountable subset X of $\mathbb{P}$, one fills up the measure by countably many sets A with $[A]$ in X and apply the countable case.

Let $\mathbb{Q}_0$ denote denote the set of all Borel subsets of 2^ω of positive measure. Random forcing $\mathbb{Q}$ is defined $\mathbb{Q}_0/\sim$. Let $[A]$ denote the equivalence class of A.

Suppose that M is a transitive model of ZFC. Let $\mathbb{P}^M \in M$ denote random forcing as defined in M. We will simply write $\mathbb{P}$ for $\mathbb{P}^M$. Suppose that G is $\mathbb{P}$-generic over M. Let

$$x_G = \bigcup_{N_t \in G} t.$$

It is easy to see that $x_G\colon \omega \to 2$ is a well-defined function. For example, it is total since for each $n \in \omega$, the set $\{N_t \mid |t| = n\}$ is dense in $\mathbb{P}$.

x_G is called a random real over M.

We claim that x_G generates $M[G]$ (as for Cohen forcing). Let $x = x_G$. How can we reconstruct G from x? Let

$$G_x := \{A \in \mathbb{P} \mid \lim_{n\to\infty} \frac{\mu(A \cap N_{x\restriction n})}{\mu(N_{x\restriction n})} = 1\}.$$

I.e. $A \in G_x$ if and only if x is a *density point* of A. Clearly G_x is a filter. Note that by Lebesgue's density theorem, almost all elements of A are density points of A.

Note that in general, every $\mathbb{P}$-generic filter G over M is maximal. To see this, suppose that there exists some $A \in \mathbb{P} \setminus G$ such that $H = G \cup \{A\}$ generates a filter. Working in M, let $X \subseteq \mathbb{P}$ be a maximal antichain in $\mathbb{P}$ with $A \in X$. Since G is $\mathbb{P}$-generic over M, pick some $B \in G \cap X$. Since X is an antichain and H is a filter, it follows that $A = B \in G$.

Claim. $G = G_x$.

Proof. It suffices to show that $G \subseteq G_x$. Towards a contradiction, suppose that for some $A \in G$, $\delta := \lim_{n\to\infty} \frac{\mu(A\cap N_{x\restriction n})}{\mu(N_{x\restriction n})} < 1$. Pick some ϵ with $\delta < \epsilon < 1$.

Pick some n_0 such that for all $n \geq n_0$, $\frac{\mu(A\cap N_{x\restriction n})}{\mu(N_{x\restriction n})} < \epsilon$. Let $t_0 := x\restriction n_0$. Since $A, N_{t_0} \in G$, we have $A' := A \cap N_{t_0} \in G$.

Now let

$$D := \left\{B \in \mathbb{P} \mid B \cap A' = \emptyset \vee \left(\exists t \supseteq t_0 \ (B \subseteq A' \cap N_t \wedge \frac{\mu(B)}{\mu(N_t)} > \epsilon)\right)\right\}$$

D is dense in $\mathbb{P}$ by Lebesgue's density theorem.[12] Pick some $B \in G \cap D$. Since $A', B \in G$, we have $B \subseteq A'$ and there exists some $t \in 2^{<\omega}$ as in the definition of D. We have $N_t \in G$, since $B \in G$.

[12]It suffices to find a single density point.

We have $\frac{\mu(B)}{\mu(N_t)} > \epsilon$. Since $B \subseteq A \cap N_t$, $\frac{\mu(A \cap N_t)}{\mu(N_t)} > \epsilon$. But this contradicts the choice of n_0 and t_0. □

3.16. **Characterising random reals.** Suppose that M is a transitive model of ZFC.

As for Cohen reals, one can show (see e.g. [Jech: Set Theory, Lemma 26.4):

Lemma 3.29. *The following are equivalent for a real x:*

(a) *x is random over M.*
(b) *$x \in B_a$ for every Borel code $a \in M$ with $\mu(B_a) = 1$.*

3.17. **Cohen reals versus random reals.** We can separate Cohen and random extensions as follows.

There exists a partition $2^\omega = A \cup B$[13] such that A has measure 1 and B is comeager. To see this, it suffices to find a dense open (hence comeager) subset C_n of 2^ω of measure $\leq \frac{1}{2^n}$ for any $n \in \omega$. Then $C := \bigcap_{n \in \omega} C_n$ is a comeager null set.

We construct C_n as follows. Let $\langle q_i \mid i \in \omega \rangle$ enumerate a countable dense subset of 2^ω. Let U_i be an open set containing q_i with $\mu(U_i) < \epsilon_i$, where $\sum_{i \in \omega} \epsilon_i \leq \frac{1}{2^n}$. Let

$$C_n := \bigcup_{i \in \omega} U_i$$

It follows that a Cohen real over M is never random over M and conversely.

One can in fact separate Cohen extensions from random extensions in the following sense: a Cohen extension $M[G]$ is never equal to a random extension $M[H]$. For example, one can show that a Cohen real over M does not add any random real over M and conversely. One can also show that the set of ground model reals has different properties in Cohen versus random extensions.

[13] A partition means that $2^\omega = A \cup B$ and A and B are disjoint.