Forcing over choiceless models (3/4)

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Outline

0. Introduction
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    - Preliminaries
    - Cohen's first model and Dedekind finite sets A
    - Properties of $\mathrm{Add}(\kappa, 1)$ and fragments of DC
    - Adding Cohen subsets over $L(\mathbb{R})$
2. Chain conditions and cardinal preservation
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    - An iteration theorem
    - A ccc_2 forcing that collapses ω_1
3. Generic absoluteness principles inconsistent with choice
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4. Random algebras without choice
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Generic diversity

Sentences that can be forced true or false over any model of ZFC:

- CH
- $b \geq \omega_2$
- A Suslin tree exists

Sentences that can be forced true and remain true in further extensions:

- There exists a non-constructible real
- ω_4^L is countable

Sentences that hold in all generic extension assuming large cardinals:

- Any sentence in $L(\mathbb{R})$
- Any sentence in the Chang model $L(\mathrm{Ord}^\omega)$

Generic diversity

Given at least one regular uncountable cardinal κ, one can force some non-trivial statements.

- ω_1 is regular
- $b = d = \kappa$
- Fragments of Martin's axiom

But there might be no uncountable regular cardinals.

Generic diversity

$\mathbb{R}$ denotes random forcing. It consists of all Borel codes for subsets of 2^ω. The quasi-order on $\mathbb{R}_\alpha$ is given by inclusion.

Fact

A random extension $V[x]$ of V satisfies ω^ω-bounding, while a Cohen extension does not.

In particular, Cohen and random extensions are different. Truss proved the following stronger statement: Cohen and random forcing don't commute.

Theorem (Truss 1983)

A $\mathbb{R} * \dot{\mathbb{C}}$-extension of V is not a $\mathbb{C} * \dot{\mathbb{R}}$-extension of V.

"Every uncountable subset B of ω_1 contains an infinite subset $A \in V$"
holds in the former, but not the latter.

We use the special case:

Fact

If y is random over $V[x]$, then x is not Cohen over $V[y]$.

Proof (Glazer, ?). Otherwise $x + y$ is both random over $V[x]$ and Cohen over $V[y]$. Then $x + y$ is both random and Cohen over V, contradiction. □

Generic diversity

$\mathbb{R}_\alpha$ denotes the random algebra on α. It consists of all Borel codes for subsets of 2^α. The quasi-order on $\mathbb{R}_\alpha$ is given by inclusion.

- $\mathbb{R}_\alpha$ is not equivalent to the finite support product of random forcings.
- We will see that $\mathbb{R}_\alpha$ preserves all cardinals.

Definition

- A Cohen model is a $\mathbb{C}^\kappa$-extension over V for some $\kappa \geq \omega_2$.
- A random model is a $\mathbb{R}_\kappa$-extension over V for some $\kappa \geq \omega_2$.

Proposition (Woodin)

Cohen and random models over V have different theories.

Proof. In a Cohen model, for any subset A of ω_1 there is a Cohen real over $V[A]$ and hence over $L[A]$, since A is constructed from an ω_1 size piece of the generic.

In the random model, let B be a piece of the random generic of size ω_1. Then there is no Cohen real x over $L[B]$.

To see this, note that for any real x, B adds a random real y over $V[x]$ and hence over $L[x]$, since x is constructed from a countable piece of B. So x is not Cohen over $L[y]$. □

The next step is to separate the theories of other extensions.

Definition

- A Hechler model is an extension of V by an iteration of Hechler forcing of length $\kappa \geq \omega_2$.

Problem

Do Cohen and Hechler models have different theories?

Generic diversity

Proposition (Aspero, Karagila 2020)
The Chang model cannot have generic absoluteness for its Σ_2 theory in ZF, even in the presence of large cardinals.

Proof sketch. Suppose that V is a model of ZFC with large cardinals.

- Form a symmetric extension M of V such that $M \models \mathrm{cof}(\omega_2) = \omega_1$ via $\mathrm{Col}(\omega_1, {<}\aleph_{\omega_1})$. Then M has the same Chang model $L(\mathrm{Ord}^\omega)$ as V.
- Let G be $\mathrm{Col}(\omega, \omega_1)$-generic over M. $M[G]$ collapses ω_1^M and $M[G] \models \mathrm{cof}(\omega_1) = \omega$.

But $\mathrm{cof}(\omega_1) = \omega$ is a Σ_2 statement over the Chang model. □

Generic diversity

If κ is supercompact in V, then κ is supercompact in M in the following sense for all α:

Definition

κ is V_α-supercompact if for every α, there exists some $\beta > \alpha$ and an elementary embedding $j\colon V_\beta \to N$ with $\alpha < \mathrm{crit}(j) = \kappa$ such that N is a transitive set with $N^{V_\alpha} \subseteq N$.

Problem (Aspero, Karagila 2020)

- *Can generic absoluteness for $L(\mathbb{R})$ fail in the presence of large cardinals?*
- *Is it possible that $\mathbb{R}^\#$ exists and ω_1 is singular in $L(\mathbb{R})$?*

□

Definition

Suppose that $\lambda < \kappa$ are cardinals.

- κ is a λ-strong limit if for all $\nu < \kappa$, $\kappa \not\leq^* \nu^\lambda$.
- κ is called λ-inaccessible if it is a λ-strong limit and $\mathrm{cof}(\kappa) > \lambda$.

Hartog numbers

Let $\aleph(x)^-$ denote $\aleph(x)$ if this is a limit cardinal and its cardinal predecessor otherwise.

We write

$$\aleph := \aleph(2^\omega) = \sup\{\alpha \in \mathrm{Ord} \mid \alpha \leq 2^\omega\},$$

$$\aleph^- := \aleph(2^\omega)^-.$$

Then

$$\aleph^- = \sup\{\lambda \in \text{Card} \mid \alpha \leq 2^\omega\}.$$

Case

$\aleph = \kappa^+$. Then $\kappa = \sup\{\lambda \in \text{Card} \mid \lambda \leq 2^\omega\}$.

Case

$\aleph$ is a limit. Then $\aleph = \sup\{\lambda \in \text{Card} \mid \lambda \leq 2^\omega\}$.

Lemma

$\aleph(\kappa^\omega) = \aleph^{V[G]}$ *for any infinite cardinal* κ *and any* $\mathbb{C}^\kappa$*-generic filter* G *over* V*.*

Proof. $\leq$: It suffices to show $\kappa^\omega \leq (2^\omega)^{V[G]}$.

- Map κ^ω injectively to a subset of κ^ω of functions with almost disjoint ranges.
- For each range, glue the list of Cohen reals into a single real. The reals are pairwise different.

$\geq$: Suppose $1 \Vdash \vec{x} = \langle \dot{x}_\alpha \mid \alpha < \gamma \rangle$ is injective. Working in $\mathrm{HOD}_{\vec{x},\Vdash}$, we can replace each $\dot{x}_\alpha$ by a nice name coded by an element of κ^ω. $\square$

Hartog numbers

Lemma

$1_{\mathbb{C}^\kappa} \Vdash \aleph = \kappa^+$ *for any ω-inaccessible cardinal κ.*

Proof. The claim is equivalent to $\aleph(\kappa^\omega) = \kappa^+$ by the previous lemma.

Otherwise there exists an injective function $f\colon \kappa^+ \to \kappa^\omega$.

- $\kappa^\omega = \bigcup_{\alpha<\kappa} \alpha^\omega$, since $\operatorname{cof}(\kappa) > \omega$.
- $|f^{-1}[\alpha^\omega]| \geq \kappa$ for some $\alpha < \kappa$.

This contradicts that κ is an ω-strong limit. □

Corollary

Suppose there exist two uncountable regular cardinals $\kappa < \lambda$. Then we can force two different theories.

Proof. Suppose that $\kappa < \lambda$ are least. Pick ω-inaccessibles ν_κ and ν_λ with cofinalites κ and λ. Then

- $1_{\mathbb{C}^{\nu_\kappa}} \Vdash \operatorname{cof}(\aleph^-) = \kappa$.
- $1_{\mathbb{C}^{\nu_\lambda}} \Vdash \operatorname{cof}(\aleph^-) = \lambda$. □

Hartog numbers

Suppose there exists only a single uncountable regular cardinal κ.

Woodin proved that one can still force two different theories via $\mathbb{C}^\lambda$ for different λ. Next is a version of this argument.

Definition

Suppose that I and J are subsets of ν^ω.

1. J covers I if for each $f \in I$, there exists some $g \in J$ with $\mathrm{ran}(f) \subseteq \mathrm{ran}(g)$.
2. For any cardinal ν, a subset J of ν^ω of size $\aleph^-$ is called minimal if it is not covered by any subset I of ν^ω of size $<\aleph^-$.
3. m denotes the least cardinal ν such that there exists a minimal subset of ν^ω, if there exists such a ν.

The idea is to find different values of m in $\mathbb{C}^\lambda$-extensions.

Lemma

$1_{\mathbb{C}^\kappa} \Vdash \mathrm{m} \geq \kappa$ *for any ω-strong limit cardinal κ.*

Proof. Work in a $\mathbb{C}^\kappa$-generic extension of V. We work in $V[G]$.

Suppose that $\nu < \kappa = \aleph$ and B is a subset of ν^ω of size κ. We claim that B is not minimal.

It suffices to find a wellorderable subset $A \in V$ of ν^ω that covers B. Since κ is an ω-strong limit in V, $|A| < \kappa$ follows.

- Fix a bijection $f\colon \kappa \to B$ and a name $\dot{f}$ for it. Let $\dot{g}$ be a $\mathbb{C}^\kappa$-name for the function $g\colon \kappa \times \omega \to \nu$ with $g(\alpha, n) = f(\alpha)(n)$. Let p force the above for $\dot{f}$ and $\dot{g}$.
- For each $(\alpha, n) \in \kappa \times \omega$, let $D_{\alpha,n}$ denote the set of all conditions $\leq p$ in $\mathbb{C}^\kappa$ that decide $\dot{g}(\alpha)(n)$. Define $g_{\alpha,n}\colon D_{\alpha,n} \to \nu$ such that $g_{\alpha,n} = \gamma$ if $r \Vdash \dot{g}(\alpha)(\beta) = \gamma$.

Then $\mathrm{ran}(g_{\alpha,n})$ is countable. Working in $\mathbf{HOD}_{\mathbb{C}^\kappa, \Vdash, \dot{f}, \dot{g}}$, we can define $h\colon \kappa \times \omega \to \nu^\omega$ such that $h(\alpha, n)$ is an enumeration of $\mathrm{ran}(g_{\alpha,n})$.

Let $\bar{h}\colon \alpha \to \nu^{\omega\times\omega}$, $\bar{h}(\alpha)(m, n) = h(\alpha, m)(n)$. Then $\bar{h}(\alpha)$ covers $f(\alpha)$. $\square$

Lemma

Suppose that $\nu \in \mathrm{Card}$*,* $p \in \mathbb{P}_\nu$ *forces that* $\aleph$ *is a successor cardinal and* $1_{\mathbb{P}} \Vdash \aleph > (\aleph^+)^V$.

Then $p \Vdash_{\mathbb{C}^\nu} \mathfrak{m} \leq \nu$.

Proof. Let $\lambda := (\aleph^-)^{V[G]} = \aleph(\nu^\omega)^-$. Then $\lambda \leq \nu^\omega$.

We claim that any subset of ν^ω of size λ in V is minimal in $V[G]$.

Fix an injective function $f\colon \lambda \to \nu^\omega$ in V.

- If $\mathrm{ran}(f)$ is not minimal, then there exists some $\mu < \lambda$, a $\mathbb{C}^\nu$-name $\dot{g}$ for a function $\dot{g}\colon \mu \to \nu^\omega$ such that some $q \leq p$ forces that $\mathrm{ran}(\dot{g})$ covers $\mathrm{ran}(f)$.
- Like in the previous proof, replace $\dot{g}$ by a function $h\colon \mu \to \nu^\omega$ in V such that $\mathrm{ran}(h)$ covers $\mathrm{ran}(f)$.

For each $\alpha < \mu$, let $A_\alpha := \{\gamma < \lambda \mid f(\gamma) \subseteq h(\alpha)\}$.

- Since $h(\alpha)$ is countable, $\mathrm{otp}(A_\alpha) < \aleph^V$ for all $\alpha < \mu$.
- We have $\bigcup_{\alpha<\mu} A_\alpha = \lambda$ since $\mathrm{ran}(h)$ covers $\mathrm{ran}(f)$, contradicting $\lambda \geq (\aleph^+)^V$.

□

Definition

Let $\mathbb{C}^*$-absoluteness ($A_{\mathbb{C}^*}$) be the statement that for any cardinal κ, the $\mathbb{C}^\kappa$-generic extension has the same theory as V.

Theorem

If $A_{\mathbb{C}^*}$ holds, then $\mathbf{1}_{\mathbb{C}^\kappa} \Vdash \aleph > \kappa^+$ for any ω-strong limit cardinal κ.

Proof. Towards a contradiction, suppose that there exists an ω-strong limit cardinal κ with $p \Vdash_{\mathbb{P}_\kappa} \aleph = \kappa^+$ for some $p \in \mathbb{P}_\kappa$. By the above, $p \Vdash_{\mathbb{C}^\kappa} \mathrm{m} \geq \aleph^-$.

It suffices to show that $\mathrm{m} < \aleph^-$ holds in a $\mathbb{C}^\lambda$-generic extension for some $\lambda \in \mathrm{Card}$.

To see this, pick any successor cardinal $\lambda \geq \aleph^+$. Since $\mathbb{C}^\kappa$ forces that $\aleph$ is the successor of a limit, the same holds for $\mathbb{C}^\lambda$ by $A_{\mathbb{C}^*}$.

Since λ is not a limit cardinal, $\mathbf{1}_{\mathbb{P}_\lambda} \Vdash \aleph > \lambda^+$.

Since $\mathbf{1}_{\mathbb{P}_\lambda}$ forces that $\aleph$ is a successor, $\mathbf{1}_{\mathbb{P}_\lambda}$ forces $\mathrm{m} \leq \lambda < \aleph^-$ by the previous Lemma. □

Corollary (Woodin)

If there exist a uncountable regular cardinal, then $A_{\mathbb{C}^*}$ fails. Then there exists an ω-inaccessible cardinal κ and we get both $\mathbf{1}_{\mathbb{C}^\kappa} \Vdash \aleph = \kappa^+$ and $\mathbf{1}_{\mathbb{C}^\kappa} \Vdash \aleph > \kappa^+$.

Gitik's model

It is open whether $A_{\mathbb{C}^*}$ is consistent. A model of $A_{\mathbb{C}^*}$ would not have uncountable regular cardinals.

Theorem (Gitik 1980)

Suppose that V is a model of BG *with a global wellorder and a proper class of strongly compact cardinals, but no regular limit of strongly compact cardinals.*

Then there is a symmetric class extension $V(G)$ of V such that:

- $V(G) \models \mathrm{ZF}$.
- *In $V(G)$, every infinite cardinal has countable cofinality.*

Theorem (Busche, Schindler)

The consistency strength of the theory ZF *and "every infinite cardinal has countable cofinality" is at least* ZFC *with infinitely many Woodin cardinals.*

Gitik's model

Gitik's model is constructed as a symmetric extension $V(G)$ of V.

The forcing $\mathbb{P}$ is constructed from a sequence of interleaved strongly compact Prikry forcings.

- Let $\langle \kappa_i \mid i \in \mathrm{Ord} \rangle$ list all strongly compact cardinals in V. Its closure equals the class of uncountable cardinals in $V(G)$.
- I is a class of finite subsets of Reg with a closure property. Each $s \in I$ induces a subforcing $\mathbb{P}_s$ of $\mathbb{P}$.

Lemma (Gitik)

For any set of ordinals $X \in V(G)$, there exists some $s \in I$ with $X \in V[G \upharpoonright \mathbb{P}_s]$.

Lemma (Gitik)

For any $s \in I$ and any strongly compact $\kappa_i \in s$, $\mathbb{P}_s$ is equivalent to a forcing $\mathbb{P}_{s \cap \kappa_i} * \dot{\mathbb{Q}}$, where $\mathbb{P}_{s \cap \kappa_i}$ forces that $\dot{\mathbb{Q}}$ does not add any bounded subsets of κ_i.

Gitik's model

Proposition

In Gitik's model, $\aleph(2^\kappa) = \kappa^+$ for all infinite cardinals κ.

Proof. Suppose that κ is an infinite cardinal in $V(G)$ and $f\colon \gamma \to {}^\omega\kappa$ is an injective function in $V(G)$. It suffices to show $\gamma < (\kappa^+)^{V(G)}$.

$\kappa = \kappa_\zeta$ and $(\kappa^+)^{V(G)} = \kappa_\xi$ for some $\zeta < \xi$, where κ_i is the ith strongly compact cardinal in V.

By the above properties of Gitik's construction, there exists some $s \in I$ with $f \in V[G{\restriction}\mathbb{P}_s]$. We may assume $\kappa_\zeta, \kappa_\xi \in s$.

Let λ be inaccessible in V with $\max(s \cap \kappa_\xi) < \lambda < \kappa_\xi$. Then:

- $\mathbb{P}_s$ is equivalent to a forcing of the form $\mathbb{P}_{s\cap\kappa_\xi} * \dot{\mathbb{Q}}$, where $\mathbb{P}_{s\cap\kappa_\xi}$ forces that $\dot{\mathbb{Q}}$ does not add new bounded subsets of κ_ξ.
- Since $|\mathbb{P}_{s\cap\kappa_\xi}| < \lambda$, λ remains inaccessible in $V[G{\restriction}\mathbb{P}_s]$.
- Since $f \in V[G{\restriction}\mathbb{P}_s]$ and $\kappa < \lambda$, we have $\gamma < \lambda < \kappa_\xi = (\kappa^+)^{V(G)}$. □

Gitik's model

We have seen that in Gitik's model, $\aleph(2^\kappa) = \kappa^+$ for all infinite cardinals κ. Hence $1_{\mathbb{C}^\kappa} \Vdash \aleph = \kappa^+$.

One can change the theory of Gitik's model by forcing:

- Otherwise for any ω-strong limit cardinal κ, $1_{\mathbb{C}^\kappa} \Vdash \aleph > \kappa^+$.
- Since $\aleph(2^\omega) = \omega_1$, no cardinal characteristics of the reals exist. But $\mathbb{C}^\kappa$ forces $b \geq \omega_1$ by $\mathbb{C}^\kappa$ for any uncountable cardinal κ.

Remark

One can show $\mathbb{C}^\kappa$ forces $b = \omega_1$ for all uncountable κ. It is open whether one can force $b \geq \omega_2$.

Similarly, one can show $\mathbb{C}^\kappa$ forces "*d* does not exist". It is open whether one can force "*d* exists".

Problem

What else can you force over Gitik's model?

Problem

Is the theory of Gitik's model the same when leaving out some strongly compact cardinals?

Problem

Is $\mathbb{C}^\kappa$-generic absoluteness consistent?

Very different properties than those of Gitik's model are needed.