# Forcing over choiceless models (1/4)

Philipp Schlicht

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# Outline

We explore properties of forcing over models of ZF such as ccc and closure and their consequences.

This addresses a question asked by Asaf Karagila on mathoverflow (2012):

> "I am looking for theorems such as c.c.c. forcing does not collapse cardinals and similar theorems extended to the choiceless context if possible, or the strength of choice needed for these theorems to hold."

Several results are folklore and various joint with Daisuke Ikegami (not attributed). Others by Woodin, Cunningham, Trang, Karagila and Schweber.

Detailed notes will be posted.

# Outline

0. Introduction
1. Adding Cohen subsets by $\mathrm{Add}(A, 1)$
    - Preliminaries
    - Cohen's first model and Dedekind finite sets $A$
    - $\mathrm{Add}(\kappa, 1)$ and fragments of DC
    - Adding Cohen subsets over $L(\mathbb{R})$
2. Chain conditions and cardinal preservation
    - Variants of the ccc
    - An iteration theorem
    - A $\mathrm{ccc}_2$ forcing that collapses $\omega_1$
3. Generic absoluteness principles inconsistent with choice
    - Hartog numbers
    - Very strong absoluteness and consequences
    - Gitik's model
4. Random algebras without choice
    - Completeness
    - $\mathrm{ccc}_2^*$

# Introduction

# Mathematics without choice

Set theory without the axiom of choice means the axiom system ZF. One can do quite some mathematics in ZF:

- Analysis: Many classical theorems such as the intermediate value theorem
- Algebra: Results about countable groups and fields, for instance every countable field has an algebraic completion
- Logic: Most results studied in second order arithmetic and reverse mathematics
- Set theory: Transfinite induction and recursion

# Mathematics without choice

On the other hand, many important theorems do not work without choice:

- Measure theory: $\sigma$-additivity of Lebesgue measure
- Functional analysis: The Hahn-Banach theorem
- Algebra: Existence of maximal ideals in rings
- Logic: Existence of nontrivial ultrafilters on the natural numbers
- Set theory: Existence of uncountable regular cardinals

# Mathematics without choice

However, some models of set theory without choice have huge advantages.

The game $G(X)$ for a subset $X$ of $\omega^\omega$ is played by players A and B.
They play natural numbers $n_i$ in turn. A wins if $\langle n_i \mid i \in \omega \rangle \in X$.

## Definition

The Axiom of Determinacy (AD) states that $G(X)$ is determined for all $X$, i.e., A or B has a winning strategy.

- Many difficult problems about projective sets, those definable over the real numbers, were solved using AD.
- AD implies that all sets of reals are Lebesgue measurable.

# Mathematics without choice

The canonical model of AD is $L(\mathbb{R})$, the constructible universe over the reals.

- $L_0 := \mathrm{tc}(\{\mathbb{R}\})$
- $L_{\alpha+1}(\mathbb{R}) := \mathrm{Def}(L_\alpha(\mathbb{R}))$
- $L_\gamma(\mathbb{R}) := \bigcup_{\alpha<\gamma} L_\alpha(\mathbb{R})$ for limits $\gamma$

# Forcing without choice

Forcing over $L(\mathbb{R})$ has been used for various reasons.

1. Forcing over models of determinacy
   - Introduced by Steel and Van Wesep (1982)
   - Powerful machinery of Woodin's $\mathbb{P}_{\max}$-forcing and variants
2. Preserving determinacy by forcing
   - Work of Chan, Jackson (2021), Ikegami and Trang (2023)
3. Combinatorics of ultrafilters on $\omega$:
   - Blass (1988) proved that an ultrafilter on $\omega$ is Ramsey if and only if it is generic over $L(\mathbb{R})$ for the forcing $P(\omega)/\mathrm{fin}$, after a Levy collapse
   - Work of Laflamme, Todorcevic and others.
4. Geometric set theory of Larson and Zapletal (2020)
   - Separating consequences of choice

# Forcing without choice

Forcing over other specific choiceless models besides $L(\mathbb{R})$ has been studied.

- Monro (1983) studied forcing over Cohen's first model and preservation of fragments of the axiom of choice.

We aim for tools that work for any model of ZF.

In particular when the Axiom of Dependent Choice (DC) fails. It fails for example in these models:

- Cohen's first model
- Gitik's model (1980) where all uncountable cardinals have countable cofinality. .

# Forcing without choice

In any generic extension of Gitik's model, all uncountable cardinals have countable cofinality and hence $AC_\omega$ fails.

So one cannot force AC.

Note that every infinite cardinal having countable cofinality is the extreme opposite to models with large cardinals.

But this property has large cardinal strength.

# Forcing without choice

The following results characterise the possibility of forcing choice:

**Theorem (Blass 1979)**

*TFAE*

1. $\exists S\ \forall X\ \exists g\colon S \times \mathrm{Ord} \to X$ *is surjective (small violations of choice SVC).*
2. *There exists a forcing* $\mathbb{P}$ *such that* $1_{\mathbb{P}}$ *forces* AC*.*

**Theorem (Usuba 2022)**

*TFAE*

1. *There exists an inner (i.e., transitive class) model* $M$ *of* ZFC *and a set* $X$ *such that* $V = M(X)$*, where* $M(X)$ *denotes the least transitive model* $N$ *of* ZF *with* $M \subseteq N$ *and* $X \in N$*.*
2. *There exists a forcing* $\mathbb{P}$ *such that* $1_{\mathbb{P}}$ *forces* AC*.*

# Forcing without choice

**Theorem (Karagila 2017)**

*If $x$ is a Cohen real over $L$, then there is an intermediate model $L \subseteq M \subseteq L[x]$, the* Bristol model, *that is not of the form $L(X)$ for a set $X$.*

It follows from Usuba's result that choice cannot be forced over $M$.

# Part 1. Adding Cohen subsets

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# Preliminaries

Suppose that $A$ and $B$ are sets.

- $A \leq B$ means there exists an injective function $f\colon A \to B$.
- $A \leq^* B$ means there exists a surjective function $g\colon B \to A$.
- It is not known if $A \leq^* B \Leftrightarrow A \leq B$ for all sets implies the axiom of choice (partition problem).

All cardinals are ordinals.

- Hartog's number $\aleph(x)$ is the least ordinal $\alpha$ with $\alpha \not\leq x$.
- Lindenbaum's number $\aleph^*(x)$ is the least ordinal $\alpha$ with $\alpha \not\leq^* x$.

## Exercise

Prove that $\aleph(x)$ and $\aleph^*(x)$ exist for every set $x$ and $\aleph(x) \leq \aleph^*(x)$.

# Preliminaries

A tree $T$ is a set of sequences, closed under initial segments.

- $T$ is $<\delta$-closed if every strictly increasing sequence in $T$ of length $\alpha < \delta$ has an upper bound in $T$.

## Definition

1. Suppose that $\kappa \in \text{Card}$. The axiom of choice $\mathsf{AC}_\kappa$ for families of size $\kappa$ states:

   *For any $F\colon \kappa \to V$ with $F(\alpha) \neq \emptyset$ for all $\alpha < \kappa$, there exists a function $f\colon \kappa \to V$ with $f(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.*

2. Suppose that $\delta \in \mathrm{Ord}$ and $A$ is a class. The axiom of dependent choice $\mathsf{DC}_\delta(A)$ for sequences of length $\delta$ states:

   *Any $<|\delta|$-closed tree $T$ without end nodes on $A$ has a branch of length $\delta$.*

   $\mathsf{DC}_\delta$ denotes $\mathsf{DC}_\delta(V)$. $\mathsf{DC}$ denotes $\mathsf{DC}_\omega$.

   $\mathsf{DC}_{<\delta}(A)$ and $\mathsf{DC}_{<\delta}$ are defined in the obvious way.

### Exercise

Suppose that $\gamma \in \mathrm{Ord}$ and $A$ is a set with $A^\gamma \leq^* A$. Then $\mathsf{DC}_\gamma(A) \Rightarrow \mathsf{DC}_\delta(A)$ for all $\delta < \gamma^+$. In particular for $A = 2^\gamma$.

### Definition

A set $X$ is called Dedekind finite if it is infinite and there exists no injective function $f\colon \omega \to X$.

### Exercise

DC implies that there exist no Dedekind finite sets.

**Lemma (Boolos 1974)**

DC *holds if and only if for every infinite cardinal* $\theta$, *there exists a countable elementary substructure* $M \prec V_\theta$.

**Proof sketch.** If DC holds, the usual construction of elementary substructures goes through.

Conversely, suppose that $T$ is a tree without end nodes.

- Pick $\alpha$ such that $T \in V_\alpha$ and let $M \prec V_\alpha$ be countable elementary.
- $T \cap M$ is a tree without end nodes.
- Using a bijection $\omega \to T \cap M$, find an infinite branch of $T$. □

Properness may become vacuous without DC (Aspero, Karagila).

# Preliminaries

A quasi-order $\mathbb{P} = (P, \leq)$ is a transitive reflexive relation.

- $p, q \in \mathbb{P}$ are compatible if $\exists r \leq p, q$ and otherwise incompatible.

A forcing is a quasi-order that is separative: $p \not\leq q \Rightarrow \exists r \leq p \; q \perp r$.

We fix a canonical name $\{\dot{x} \mid \dot{x} \in X\}^\bullet$ for $\{\dot{x} \mid \dot{x} \in X\}$.

Basic properties of forcing including the forcing theorem work in ZF just like in ZFC.

Note: Fullness $p \Vdash \exists x \, \varphi(x) \Rightarrow \exists \sigma \; p \Vdash \varphi(\sigma)$ implies AC (Miller).

# Preliminaries

Any forcing $\mathbb{P}$ is densely embedded into the complete Boolean algebra $\mathbb{B} := \mathbb{B}(\mathbb{P})$, the set of all regular open subsets of $\mathbb{P}$.

A $\mathbb{B}$-valued model of ZF is constructed as

- $V_0^{\mathbb{B}} = \emptyset$
- $V_{\alpha+1}^{\mathbb{B}} = \{f \colon V_\alpha^{\mathbb{B}} \to \mathbb{B}\}$
- $V_\lambda^{\mathbb{B}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbb{B}}$ for limits $\lambda$

with union $V^{\mathbb{B}}$. $V^{\mathbb{B}}$ believes it is a $\mathbb{P}$-generic extension of a ground with the theory of $V$. (These statements hold in $V^{\mathbb{B}}/U$, where $U$ is any ultrafilter on $\mathbb{B}$.)

We can therefore safely work with $\mathbb{P}$-generic filters $G$ over $V$.

### Lemma

DC *suffices to show that every* $\sigma$*-closed forcing preserves* $\omega_1$*.*

**Proof.** Suppose that $p \Vdash \dot{f}\colon \omega \to \omega_1$ is surjective.

- Find $p = p_0 \geq p_1 \geq \dots$ such that $p_n$ decides $\dot{f}(n)$ for all $n \in \omega$.
- Let $q \leq p_n$ for all $n$. Then $q$ decides all values of $\dot{f}$. □

### Exercise

Suppose that $\omega_1$ is singular. Show that

$$\mathrm{Add}(\omega_1, 1) := \{p\colon \alpha \to 2 \mid \alpha < \kappa\}$$

collapses $\omega_1$.

If $\omega_1$ is regular, but $\mathrm{DC}(2^\omega)$ fails, then $\mathrm{Add}(\omega_1, 1)$ adds new reals.

# Symmetric models

Let $\mathbb{P}$ be a notion of forcing and $\pi$ be an automorphism of $\mathbb{P}$. Then $\pi$ acts on $\mathbb{P}$-names via

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}.$$

Suppose that $\mathscr{G}$ is a group of automorphisms of $\mathbb{P}$.

- A filter of subgroups over $\mathscr{G}$ is a nonempty family $\mathscr{F}$ of subgroups of $\mathscr{G}$ closed under finite intersections and supergroups. $\mathscr{F}$ is normal if whenever $H \in \mathscr{F}$ and $\pi \in \mathscr{G}$, then $\pi H \pi^{-1} \in \mathscr{F}$ as well.
- We call $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ a symmetric system if $\mathbb{P}$ is a forcing, $\mathscr{G}$ is a subgroup of $\mathbf{Aut}(\mathbb{P})$ and $\mathscr{F}$ is a normal filter of subgroups of $\mathscr{G}$.
- $\dot{x}$ is called $\mathscr{F}$-symmetric if its stabiliser

$$\mathrm{sym}_{\mathscr{G}}(\dot{x}) := \{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\} \in \mathscr{F},$$

  and hereditarily $\mathscr{F}$-symmetric if this holds hereditarily for names in $\dot{x}$.

# Symmetric models

The class $\mathrm{HS}_{\mathscr{F}}$ denotes the class of all hereditarily $\mathscr{F}$-symmetric names. We usually omit the subscript $\mathscr{F}$.

### Lemma (Symmetry lemma)

*Suppose that $p \in \mathbb{P}$, $\pi \in \mathrm{Aut}(\mathbb{P})$ and $\dot{x}$ is a $\mathbb{P}$-name. Then*

$$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi\dot{x}).$$

### Theorem

*Suppose that $G$ is a $\mathbb{P}$-generic filter over $V$. Then*

$$M := \mathrm{HS}^G := \{\dot{x}^G \mid \dot{x} \in \mathrm{HS}\}$$

*is a transitive class model of ZF in $V[G]$ such that $V \subseteq M$.*

$\mathrm{HS}^G$ is called a symmetric extension of $V$.

# Cohen's first model

Suppose that $V$ is a model of **ZFC** and $\mathbb{P}$ is

$$\mathrm{Add}(\omega, \omega) := \{p \mid p\colon \omega \times \omega \to 2 \text{ is a finite partial function}\}.$$

The group $\mathscr{G}$ consists of all finitary permutations of $\omega$ acting on the first coordinate of $\mathbb{P}$ via

$$\pi p(n, \pi m) = p(n, m).$$

$\mathscr{F}$ is the filter of subgroups generated by $\{\mathsf{fix}(E) \mid E \in [\omega]^{<\omega}\}$, where

$$\mathsf{fix}(E) := \{\pi \in \mathscr{G} \mid \pi \restriction E = \mathrm{id}\}.$$

If $\mathsf{fix}(E) \subseteq \mathsf{sym}(\dot{x})$, then $E$ is called a support for $\dot{x}$.

For each $n \in \omega$,

$$\dot{a}_n := \{\langle p, \check{m}\rangle \mid p(m, n) = 1\}$$

is a name for the $n$th Cohen real and $\dot{A} := \{\dot{a}_n \mid n \in \omega\}^\bullet$ is a name for the set of them.

We have $\pi\dot{A} = \dot{A}$, since $\pi\dot{a}_n = \dot{a}_{\pi n}$ for all $\pi \in \mathscr{G}$. Hence $\dot{A} \in \mathrm{HS}$.

## Exercise

$1 \Vdash \dot{A}$ is Dedekind finite.

Let $M$ denote Cohen's first model.

Let $f\colon \omega \to 2$ be a finite partial function, and let $\dot{q}_f$ denote the following name:

$$\dot{q}_f = \left\{ \langle \dot{a}_n, f(\check{n}) \rangle^\bullet \mid \alpha \in \operatorname{dom}(f) \right\}^\bullet.$$

$\dot{\mathbb{Q}} := \{\dot{q}_f \mid f\colon \omega \to 2 \text{ is a finite partial function}\}^\bullet$ is a name for $\operatorname{Add}(\kappa, 1)^M$, since any $q \in \operatorname{Add}(A, 1)$ equals $q_f^G$ for some $f$.

## Proposition (Karagila, S.)

$\operatorname{Add}(A, 1)$ does not add new sets of ordinals over $M$.

**Proof.** Let $\dot{X} \in \mathrm{HS}$ be a $\mathbb{P}$-name for an $\operatorname{Add}(A, 1)$-name for a set of ordinals. Any $\pi \in \mathscr{G}$ acts on $\mathbb{P} * \operatorname{Add}(\dot{A}, \check{\kappa})^\bullet$ via

$$\pi\langle p, \dot{q}_f \rangle = \langle \pi p, \pi \dot{q}_f \rangle = \langle \pi p, \dot{q}_{f \circ \pi} \rangle.$$

We write $\langle p, \dot{q}_f \rangle \Vdash^{\mathrm{HS}} \varphi$ to mean that $p$ forces that $\dot{q}_f \Vdash \varphi$ holds in $V(A)$.

# Dedekind finite sets

**Theorem (Karagila, S.)**

$\mathrm{Add}(A, 1)$ *does not add new sets of ordinals over M.*

**Proof (cont.)**

Suppose that $\langle p, \dot{q}_f\rangle \Vdash^{\mathrm{HS}}$ "$\dot{X}$ is a set of ordinals".

Let $E$ be a support for $\dot{X}$. Then $E$ is a finite subset of $\omega$ with $\mathsf{fix}(E) \subseteq \mathsf{sym}(\dot{X})$. We can assume $\mathsf{supp}(p) = E = \mathrm{dom}(f)$.

Suppose that $\langle p_0, \dot{q}_{f_0}\rangle$ and $\langle p_1, \dot{q}_{f_1}\rangle$ are two extensions of $\langle p, \dot{q}_f\rangle$. Assume that $\mathsf{supp}(p_i) = \mathrm{dom}(f_i)$ for $i < 2$.

We claim that $\dot{X}$ is a name for a set in $M$. It suffices to show that if $p_1 \restriction E = p_2 \restriction E$, then $p_0$ and $p_1$ must agree on any statement of the form $\check{\alpha} \in \dot{X}$.

This holds since there exists an automorphism $\pi \in \mathsf{fix}(E)$ moving $\mathsf{supp}(p_0) \setminus E$ to be disjoint of $\mathsf{supp}(p_1)$, so $\langle \pi p_0, \pi\dot{q}_{f_0}\rangle$ is compatible with $\langle p_1, \dot{q}_{f_1}\rangle$ while $\pi\check{\alpha} = \check{\alpha}$ and $\pi\dot{X} = \dot{X}$.

So $\dot{X}$ only depends on the first component of the filter. □

# Dedekind finite sets

For other models and some Dedekind finite sets $A$, $\mathrm{Add}(A, 1)$ may add new reals.

For instance, Cohen's second model $N$ witnesses a failure of $\mathrm{AC}_\omega$ by a sequence $\langle F_n \mid n \in \omega \rangle$ of pairwise disjoint finite sets.

- The union $A$ of these sets is Dedekind finite and cannot be linearly ordered.
- Forcing with $\mathrm{Add}(A, 1)$ over $N$ adds a function $g \colon A \to 2$. Then

$$\{n \in \omega \mid g[F_n] = \{0\}\}$$

  is a Cohen real over $N$.

One can precisely characterise for which $A$ this happens (Karagila, S.).

# Cohen subsets

We now turn to adding Cohen subsets of cardinals. (Recall that all cardinals are wellordered.) The forcing is

$$\mathrm{Add}(\kappa, 1) := \{p\colon \alpha \to 2,\ \alpha < \kappa\}$$

ordered by reverse inclusion.

$\mathrm{Add}(\kappa, 1)$ is not $<\kappa$-closed if $\kappa$ is singular. However, for successors $\kappa = \nu^+$ it is equivalent to the $<\kappa$-closed forcing

$$\mathrm{Add}_*(\kappa, 1) := \{(p, q) \mid p \in \mathrm{Add}(\kappa, 1),\ q\colon \mathrm{dom}(p) \to \nu \text{ is bijective}\}$$

ordered by reverse inclusion in the first coordinate.

## Exercise

Show that $\mathrm{Add}(\omega_1, 1)$ wellorders the reals, and collapses $\omega_1$ if $\omega_1$ is singular.

# Cohen subsets

A forcing $\mathbb{P}$ is called $<\lambda$-distributive if for any sequence $\langle U_i \mid i < \alpha \rangle$ of dense open subsets of $\mathbb{P}$ of length $\alpha < \lambda$, $\bigcap_{i<\alpha} U_i \neq \emptyset$.

A $\lambda$-distributive forcing does not add element of $V^\lambda$. The converse implication may fail (Karagila, Schilhan).

## Lemma (folklore)

*For any infinite cardinal $\lambda$, TFAE*

- $\mathrm{DC}_\lambda$.
- *Every $<\lambda^+$-closed forcing is $<\lambda^+$-distributive.*

**Proof.** Using $\mathrm{DC}_\lambda$, we can find a sequence $\langle p_i \mid i < \lambda \rangle$ with $p_i \in U_i$ for all $i < \lambda$. Any lower bound $p$ of this sequence is in $\bigcap_{i<\lambda} U_i$.

Conversely, if $\mathrm{DC}_\lambda$ fails then there exists a $<\lambda$-closed tree $T$ with no $\lambda$-sequences, so $T$ is $<\lambda^+$-closed. Forcing with $(T, \geq)$ adds a new $\lambda$-sequence, so $T$ cannot be $<\lambda$-distributive. □

**Lemma**

*Suppose that* $\lambda \in \text{Card}$ *and* $\mathbb{P} = \mathrm{Add}(\lambda^+, 1)$. *TFAE*

1. $\mathsf{DC}_\lambda(2^\lambda)$.
2. $\mathbb{P}$ *is* $\lambda$*-distributive.*
3. $\mathbb{P}$ *does not change* $V^\lambda$.

**Proof.** $1 \Rightarrow 2 \Rightarrow 3$ are as in the previous lemma.

$3 \Rightarrow 1$: $\mathbb{P}$ wellorders $(2^\lambda)^{<\lambda}$. Thus the given tree $T$ has a $\lambda$-branch in the extension. Since $\mathbb{P}$ does not change $V^\lambda$, this branch is in $V$. □

# Cohen subsets

## Proposition

Suppose that $\lambda \in \text{Card}$ and $\mathbb{P} = \text{Add}(\lambda^+, 1)$. TFAE

1. $\text{DC}_\lambda(2^\lambda)$.
2. $\mathbb{P}$ preserves all cardinals $\alpha \leq \lambda^+$ and the cofinality of all ordinals $\alpha \leq \lambda^+$.
3. $\mathbb{P}$ preserves $\lambda^+$ as a cardinal.
4. $\mathbb{P}$ forces that $\lambda^+$ is regular.

**Proof.** $1 \Rightarrow 2$ holds by the previous lemma and $2 \Rightarrow 3$ is clear. $3 \Rightarrow 4$ holds since $\mathbb{P}$ wellorders $2^\lambda$.

$4 \Rightarrow 1$: Let $\nu \leq \lambda$ be least such that $\mathbb{P}$ adds new elements to $V^\nu$. Then $\nu$ is regular. It suffices to show that $\mathbb{P}$ is $\nu$-distributive.

Suppose that $\langle U_i \mid i < \nu \rangle \in V$ is a sequence of dense open subsets of $\mathbb{P}$ and $G$ is a $\mathbb{P}$-generic filter over $V$.

Since $\mathbb{P}$ does not change $V^{<\nu}$, construct a strictly decreasing sequence $\langle p_i \mid i < \nu \rangle$ with $p_i \in U_i \cap G$ in $V[G]$.

Since $\lambda^+$ is regular in $V[G]$ and $\nu < \lambda^+$, $p := \bigcup_{i<\nu} p_i$ is the unique condition in $G$ of length $\mu$. In particular, $p \in V$. □

# Cohen subsets

The previous tells us only whether $\lambda^+$ is preserved.

## Problem

*Suppose that* $\mathrm{Add}(\lambda^+, 1)$ *collapses* $\lambda^+$. *Which combinations of cardinals* $\leq \lambda$ *can be preserved or collapsed?*

## Problem

*Can* $\mathrm{Add}(\omega_2, 1)$ *preserve* $\omega_1$ *while* $\mathsf{DC}(2^\omega)$ *fails?*
*Is this true over Cohen's first model?*

# Cohen subsets over $L(\mathbb{R})$

We now force over $L(\mathbb{R})$.

Note that the theory of $L(\mathbb{R})$ does not change when forcing over a model $V$ of ZFC if there is a proper class of Woodin cardinals in $V$.

- In this setting, $L(\mathbb{R})$ satisfies the axiom of determinacy AD.
- In $L(\mathbb{R})$, AD implies DC by a result of Kechris.

However, we force over $L(\mathbb{R})$ and in general $L(\mathbb{R})[G] \neq L(\mathbb{R})^{V[G]}$, since $\mathbb{R}^V \notin L(\mathbb{R})^{V[G]}$ in the above situation.

# Cohen subsets over $L(\mathbb{R})$

Suppose that AD holds in $L(\mathbb{R})$.

A Cohen real $x$ over $L(\mathbb{R})$ preserves cardinals. The usual proof for ccc forcings works, since Cohen forcing is wellordered.

A Cohen subset of $\lambda^+$ over $L(\mathbb{R})$ collapses $\lambda^+$ by the previous proposition, since $\mathsf{DC}_{\omega_1}(2^\omega)$ fails

## Problem

*Which cardinals* $\leq\lambda$ *does* $\mathrm{Add}(\lambda^+, 1)$ *preserve or collapse?*

One can ask whether adding a Cohen subset preserves AD.

A Cohen real $x$ over $L(\mathbb{R})$ destroys AD (Ikegami, Trang). This follows from a result of Kunen that $\mathbb{R}^V$ doesn't have the Baire property in a Cohen extension.

**Proposition (Chan, Jackson, Goldberg 2021)**

Any well-ordered forcing in $L(\mathbb{R})$ that adds new reals destroys AD.

**Proof sketch.** In $V[G]$, take a perfect tree $T$ with $[T] \subseteq (2^\omega)^V$.

- For each $p \in \mathbb{P}$, let $A_p$ be the set of $x \in 2^\omega$ such that $p \Vdash x \in [T]$.
- Some $A_p$ is uncountable, since a wellordered union of meager sets is meager by Kuratowski-Ulam and the Baire property.
- Take a perfect tree $T'$ with $[T'] \subseteq A_p$. Then $p \Vdash [T'] \subseteq [T]$. Since $\mathbb{P}$ adds new reals, it adds a new element of $[T']$. □

A Cohen subset of $\lambda^+$ over $L(\mathbb{R})$ destroys AD, since it forces AC.

# Cohen subsets over $L(\mathbb{R})$

However, there is a useful method of forcing with $\mathrm{Add}(\kappa, 1)^{\mathsf{HOD}}$ without collapsing $\kappa$.
It uses that one can force choice by a $\sigma$-closed forcing.

## Definition

Suppose that $M$ is a transitive model and $A \subseteq \kappa$. $A$ is a fresh subset of $\kappa$ over $M$ if $A \notin M$, but $A \cap \alpha \in M$ for all $\alpha < \kappa$.

## Theorem (Cunningham 2023)

*Suppose that $V$ is a model of ZFC, $L(\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD}$ and $\kappa > |\mathbb{R}|$ is a regular cardinal.*
*If $A$ is a fresh subset of $\kappa$ over $L(\mathbb{R})$, then $L(\mathbb{R})[A]$ does not have new sets of reals and is a model of AD.*

**Proof sketch.** Suppose that $B \in L(\mathbb{R})[A]$ is a set of reals. $B$ is definable over some $L_\alpha(\mathbb{R})[A]$ from a real and an ordinal. Assume no parameters are used.

If $\alpha < \kappa$, then $L_\alpha(\mathbb{R})[A] = L_\alpha(\mathbb{R})$ since $A \subseteq \kappa$ is fresh and thus $B \in L(\mathbb{R})$.

If $\alpha \geq \kappa$, fix an operator $H$ in $V$ for Skolem hulls in $L_\alpha(\mathbb{R})$ and let

$$H_\xi := H^{L_\alpha(\mathbb{R})[A]}(\mathbb{R} \cup \{\xi\})$$

for $\xi \in \mathrm{Ord}$. Let $M_0 := H_0$, $\xi_n := M_n \cap \kappa$ and $M_{n+1} := H_{\xi_n}$.

- $M := \bigcup_{n \in \omega} M_n \prec L_\alpha(\mathbb{R})$ and $\xi := M \cap \kappa \in \kappa$, since $\kappa$ is regular.
- The transitive collapse $N$ of $M$ equals $L_\beta(\mathbb{R})[A \cap \xi]$ for some $\beta < \kappa$.
- $B \in L(\mathbb{R})[A]$, since $B$ is definable over $N$. □

# Cohen subsets over $L(\mathbb{R})$

## Theorem (Cunningham 2023)

*Suppose that $L(\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD}$ and $\kappa > (\aleph^*(\mathbb{R})^+)^{L(\mathbb{R})}$ is regular. Suppose that in $\mathsf{HOD}^{L(\mathbb{R})}$,*

$$1_{\mathbb{P}} \Vdash \text{“}\kappa \text{ is a regular cardinal”}$$

*and the $\mathbb{P}$-generic filter is a fresh subset of $\kappa$. Then $\mathbb{P}$ preserves regularity of $\kappa$ and $\mathsf{AD}$ over $L(\mathbb{R})$.*

**Proof sketch.** We assume $\kappa$ is sufficiently large. One can check that $\kappa$ remains regular in $L(\mathbb{R})[H]$.

Suppose that $G \times H$ is $\mathbb{P} \times \mathrm{Col}(\omega_1, \mathbb{R})$-generic over $L(\mathbb{R})$.

The subset $A$ of $\kappa$ given by $G$ is a fresh subset of $\kappa$ over $L(\mathbb{R})$.

By the previous theorem, it suffices to show that $\kappa$ is regular in $L(\mathbb{R})[G \times H]$.

We have $M := \mathsf{HOD}^{L(\mathbb{R})} = \mathsf{HOD}^{L(\mathbb{R})[H]}$, since $\mathrm{Col}(\omega_1, \mathbb{R})$ is homogeneous.

Let $\mathbb{Q}$ denote the Vopenka forcing in $M$ for subsets of $\omega_1$. Then the subset $B$ of $\omega_1$ given by $H$ is $\mathbb{Q}$-generic over $M$.

Since $B$ codes all reals, $L(\mathbb{R})[H] = M[H]$ is a $\mathbb{Q}$-generic extension of $M$ and

$$M[G][H] = M[H][G] = L(\mathbb{R})[G \times H].$$

By assumption, $\kappa$ remains regular in $M[G]$. If $\kappa > |\mathbb{Q}|^M$, then $\kappa$ is regular in $M[G][H]$. □

# Cohen subsets over $L(\mathbb{R})$

Ikegami and Trang (2023) improved the assumption in the previous theorem for $\kappa \geq \aleph^*(\mathbb{R})$. This is optimal by results of Chan and Jackson (2021).

Their work was motivated by the problem:

**Problem (Ikegami, Trang 2023)**

*Can there exist an elementary embedding $j\colon V \to V[G]$ with critical point $\omega_1$ such that $(V, \in, j)$ is a model of* AD*?*