A SURVEY ON FORCING OVER CHOICELESS MODELS

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Abstract. We describe known results about properties of forcings such as chain conditions and their effect on generic extensions, in particular over models where DC fails. Some of this is taken from joint work with Daisuke Ikegami.

Contents

1. Introduction

We survey known results, some folklore and some recent, about forcing over arbitrary models of ZF, in particular about adding Cohen subsets, variants of the countable chain condition and generic absoluteness. Some of them address a question asked by Asaf Karagila on mathoverflow (2012): "I am looking for theorems such as c.c.c. forcing does not collapse cardinals and similar theorems extended to the choiceless context if possible, or the strength of choice needed for these theorems to hold."

1. **Cohen subsets.** We show that forcing with $\text{Add}(A, 1)$ depends heavily on A and fragments of choice. We look at $\text{Add}(A, 1)$ for a Dedekind finite set A in Cohen's first model. We characterise when $\text{Add}(\lambda^+, 1)$ preserves λ^+ by fragments of DC. Since $\text{Add}(\lambda^+, 1)$ collapses λ^+ over $L(\mathbb{R})$, we force with $\text{Add}(\kappa, 1)^{\text{HOD}}$.
2. **Chain conditions.** We discuss variants of the ccc and whether they preserve cardinals. We prove an iteration theorem for a variant of the ccc. We discuss Karagila and Schweber's result that ccc_2 forcing can collapse ω_1.
3. **Generic absoluteness.** We study very strong generic absoluteness principles that are inconsistent with choice and their consequences. We study Gitik's model where all infinite cardinals have countable cofinality.
4. **Random algebras.** We prove that random algebras with κ many generators are complete and satisfy a version of the ccc. We show that several results above can be applied to them.

This includes work of Woodin, Cunningham [Cun23], Ikegami, Trang [IT23], Karagila and Schweber [KS22]. Besides the work mentioned here, there has been recent work on properties of forcings over arbitrary models of ZF by Karagila and Schilhan [KS23].

Forcing over models of ZF up to the forcing theorem is the same for ZFC. This can be found in [Kun14] and the Boolean-valued approach in [Jec03, HS12]. Symmetric models and Cohen's first model (the Halpern-Levy model) are studied in [Jec08].

2. Preliminaries

All models are models of ZF. A forcing is a set quasi-order $\mathbb{P}$ with $\leq$ (partial order without reflexivity) with a maximal element $1_{\mathbb{P}}$. We write $p \parallel q$ if p and q are compatible, i.e., there exists some $r \leq p, q$, and $p \perp q$ if they are incompatible. If $\mathbb{P}$ is separative, i.e., $p \nleq q$ implies $\exists r \leq p\ r \perp q$, then $\mathbb{P}$ is dense in its Boolean completion $\mathbb{B}(\mathbb{P})$, the set of regular open subsets of $\mathbb{P}$. If $\mathbb{P}$ is not separative, we form its separative quotient $\mathbb{P}_{\text{sep}}$ by letting $p \sim q$ if $\forall r\ (p \parallel r \Leftrightarrow q \parallel r)$ and work with $\mathbb{B}(\mathbb{P}_{\text{sep}})$. If $\langle x_i \mid i \in I \rangle$ is a family of $\mathbb{P}$-names, $\{x_i \mid i \in I\}^\bullet := \{(\mathbb{1}_{\mathbb{P}}, \dot{x}_i) \mid i \in I\}$ is a name for $\{\dot{x}_i \mid i \in I\}$.

For sets A and B, $A \leq B$ means there exists an injective function $f{:}\, A \to B$. $A \leq^* B$ means there exists a surjective function $g{:}\, B \to A$. The partition principle states that $A \leq^* B$ implies $A \leq B$ for all sets A and B. One of the oldest open problems in set theory asks whether the partition principle implies the axiom of choice.

A tree T on a set A consists of sequences of elements of A ordered by end extension.

Definition 2.1. Suppose that $\kappa \in \mathrm{Card}$ and $\delta \in \mathrm{Ord}$.

1. The axiom of choice AC_κ for families of size κ states that for any $F{:}\, \kappa \to V$ with $f(\alpha) \neq \varnothing$ for all $\alpha < \kappa$, there exists a choice function $f{:}\, \kappa \to V$ with $f(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.
2. Suppose that A is a class. The axiom of dependent choice $\mathsf{DC}_\delta(A)$ for trees on A states that any $<\kappa$-closed tree T on A[1] (i.e., such that every strictly increasing sequence in T of length $\alpha < \delta$ has an upper bound) has a branch of length δ. DC_δ denotes $\mathsf{DC}_\delta(V)$. Finally, $\mathsf{DC}_{<\delta}(A)$ and $\mathsf{DC}_{<\delta}$ are defined in the obvious way.
3. The axiom of determinacy AD states that any two-player game with perfect information of length ω with moves in ω is determined, i.e., one of the players has a winning strategy.

Exercise 2.2. Suppose that $\kappa \in \mathrm{Card}$, $\gamma \in \mathrm{Ord}$, $\delta < \gamma^+$ and A, B are classes.

1. DC_κ implies AC_κ.
2. $\mathsf{DC}_\lambda(A)$ implies $\mathsf{DC}_\kappa(A)$ for $\kappa \leq \lambda$.
3. $\mathsf{DC}_\gamma(A)$ for all sets A implies DC_γ.
4. If $A \leq^* B$, then $\mathsf{DC}_\gamma(B) \Rightarrow \mathsf{DC}_\gamma(A)$.
5. If $A^\gamma \leq^* A$, then $\mathsf{DC}_\gamma(A) \Rightarrow \mathsf{DC}_\delta(A)$.

Every cardinal is an ordinal.

Exercise 2.3. DC suffices to show that every σ-closed forcing preserves ω_1.

Exercise 2.4. DC holds if and only if for every infinite cardinal θ, there exists a countable elementary substructure $M \prec H_\theta$.

This shows that properness may become vacuous without DC. Aspero and Karagila showed that DC suffices to show that proper forcings have the usual properties [AK21].

A set X is *Dedekind finite* if it is infinite and there exists no injective function $f{:}\, \mathbb{N} \to X$. As usual, a set is finite if is has size n for some $n \in \mathbb{N}$.

Exercise 2.5. DC implies that there exist no Dedekind finite sets.

Remark 2.6.

1. Dedekind finite sets of reals can exist in models of ZF. If A is such as set, then linearity of $\leq$ breaks down immediately above the finite sets, since A and ω are incomparable.
2. ω_1 may be measurable, in fact AD implies the club filter on ω_1 is an $<\omega_1$-complete ultrafilter on ω_1 [Kan08].
3. ω_1 may be singular. This holds in the $L(\mathbb{R})$ of a $\mathrm{Col}(\omega, <\aleph_\omega)$-generic extension.
4. The set $\mathbb{R}$ of reals may be a countable union of countable sets. Then every set of reals is a Borel set according to the usual definition of Borel sets in ZFC [Jec08].

Exercise 2.7. Suppose that ω_1 is singular. Show that $\mathrm{Add}(\omega_1, 1) = \{p{:}\, \alpha \to 2 \mid \alpha < \kappa\}$ collapses ω_1.

[1]The height of T is arbitrary. If we consider only trees of height δ, the axiom is weaker.

Even a single regular cardinal κ allows one to do some interesting forcings. For instance, one can force the dominating number to be κ [IS22]. Models without any uncountable regular cardinals seem particularly hard to force over. Gitik constructed such a model [Git80].

Here is a bit of information about forcing choice. One cannot force choice over Gitik's model, since an end segment of cardinals in the extension has countable cofinality. Blass and Usuba characterised the possibility of forcing choice as follows.

Theorem 2.8 (Blass [Bla79])**.** *The following statements are equivalent:*

(a) $\exists S\ \forall X\ \exists g\colon S \times \mathrm{Ord} \to X$ *is surjective. This principle is called SVC (small violations of choice).*
(b) $\exists S\ \forall X\ \exists f\colon X \to S \times \mathrm{Ord}$ *is injective.*
(c) *There exists a forcing* $\mathbb{P}$ *such that* $\mathbb{1}_{\mathbb{P}}$ *forces choice.*

Theorem 2.9 (Usuba [Usu18])**.** *The following statements are equivalent:*

(a) *There exists an inner (i.e., transitive class) model* M *of* ZFC *and a set* X *such that* $V = M(X)$*, where* $M(X)$ *denotes the least transitive model* N *of* ZF *with* $M \subseteq N$ *and* $X \in N$*.*
(b) V *is a symmetric extension of some inner model* M *of* ZFC*.*
(c) *There exists a forcing* $\mathbb{P}$ *such that* $\mathbb{1}_{\mathbb{P}}$ *forces choice.*

Theorem 2.10 (Karagila [Kar18])**.** *If* x *is a Cohen real over* L*, then there is an intermediate model* $L \subseteq M \subseteq L[x]$*, the* Bristol model*, that is not of the form* $L(X)$ *for a set* X*.*

It follows from Usuba's result that choice cannot be forced over M.

3. Cohen subsets

3.1. **Cohen subsets of Dedekind finite sets.** We sketch how symmetric models are used to construct models of ZF without choice from models of ZFC. Let $\mathbb{P}$ be a notion of forcing and π be an automorphism of $\mathbb{P}$. Then π acts on $\mathbb{P}$-names via

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}.$$

Suppose that $\mathscr{G}$ is a group of automorphisms of $\mathbb{P}$. A *filter* of subgroups over $\mathscr{G}$ is a nonempty family $\mathscr{F}$ of subgroups of $\mathscr{G}$ closed under finite intersections and supergroups. $\mathscr{F}$ is *normal* if whenever $H \in \mathscr{F}$ and $\pi \in \mathscr{G}$, $\pi H \pi^{-1} \in \mathscr{F}$ as well.

We call $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ a *symmetric system* if $\mathbb{P}$ is a notion of forcing, $\mathscr{G}$ is a subgroup of $\mathrm{Aut}(\mathbb{P})$, and $\mathscr{F}$ is a normal filter of subgroups over $\mathscr{G}$. $\mathrm{sym}_{\mathscr{G}}(\dot{x})$ denotes the group $\{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\}$, the stabiliser of $\dot{x}$. $\dot{x}$ is called *$\mathscr{F}$-symmetric* if $\mathrm{sym}_{\mathscr{G}}(\dot{x}) \in \mathscr{F}$. If is called *hereditarily* $\mathscr{F}$-symmetric if this holds hereditarily for the names in $\dot{x}$. The class $\mathsf{HS}_{\mathscr{F}}$ denotes the class of all hereditarily $\mathscr{F}$-symmetric names. We usually omit the subscript $\mathscr{F}$.

Lemma 3.1 (Symmetry lemma [Jec03, Lemma 14.37])**.** *Suppose that $p \in \mathbb{P}$, $\pi \in \mathrm{Aut}(\mathbb{P})$ and $\dot{x}$ is a $\mathbb{P}$-name. Then*

$$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi\dot{x}).$$

Theorem 3.2 ([Jec03, Lemma 14.37])**.** *Suppose that $G \subseteq \mathbb{P}$ is a V-generic filter and $M := \mathsf{HS}^G := \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}\}$. Then M is a transitive class model of* ZF *in $V[G]$ such that $V \subseteq M$.*

HS^G is called a *symmetric extension* (of V).

We briefly describe Cohen's first model as an example of a model of set theory where the axiom of choice fails. It is described in detail in [Jec08, Section 5.3]. Suppose that V is a model of ZFC and $\mathbb{P}$ is $\mathrm{Add}(\omega,\omega)$. The group $\mathscr{G}$ consists of all finitary permutations of ω acting on the first coordinate of $\mathbb{P}$ via

$$\pi p(\pi n, m) = p(n, m).$$

Moreover, $\mathscr{F}$ is the filter of subgroups generated by $\{\mathrm{fix}(E) \mid E \in [\omega]^{<\omega}\}$, where $\mathrm{fix}(E) := \{\pi \in \mathscr{G} \mid \pi \restriction E = \mathrm{id}\}$. If $\mathrm{fix}(E) \subseteq \mathrm{sym}(\dot{x})$, we say that E is a *support* for $\dot{x}$.

For each $n \in \omega$, $\dot{a}_n := \{\langle p, \check{m}\rangle \mid p(n,m) = 1\}$ is a name for the nth Cohen real and $\dot{A} := \{\dot{a}_n \mid n \in \omega\}^\bullet$ is a name for the set of them. We have $\pi\dot{A} = \dot{A}$, since $\pi\dot{a}_n = \dot{a}_{\pi^{-1}n}$ for all $\pi \in \mathscr{G}$. Hence $\dot{A} \in \mathsf{HS}$.

Proposition 3.3. $\mathbb{1} \Vdash \dot{A}$ *is Dedekind finite.*

Proof. Suppose that $\dot{f} \in \mathsf{HS}$ and $p \Vdash \dot{f}{:}\check{\omega} \to \dot{A}$. Let E be a support for $\dot{f}$, and without loss of generality $\mathrm{supp}(p) \subseteq E$ as well.

We claim that p forces that the range of $\dot{f}$ is a subset of $\{\dot{a}_n \mid n \in E\}$ and hence finite. To see this, pick some $n \notin E$. Suppose towards contradiction that $q \le p$ is a condition such that $q \Vdash \dot{f}(\check{m}) = \dot{a}_n$ for some $m < \omega$. Let $j \notin E \cup \mathrm{supp}(q)$ and π the 2-cycle $(n\ j)$. Then the following statements hold:

1. $\pi \in \mathrm{fix}(E)$ and therefore $\pi p = p$ and $\pi\dot{f} = \dot{f}$.
2. $\pi\dot{a}_n = \dot{a}_j$.
3. $\pi q \Vdash \pi\dot{f}(\pi\check{m}) = \pi\dot{a}_n$ and therefore $\pi q \Vdash \dot{f}(\check{m}) = \dot{a}_j$.
4. πq is compatible with q.

The last claim holds since $j \notin \mathrm{supp}(q)$ and π only swaps the coordinates j and n. Thus, $q \cup \pi q \Vdash$ "$\dot{a}_n = \dot{f}(\check{m}) = \dot{a}_j$". This is impossible, since $\mathbb{1}_{\mathbb{P}} \Vdash \dot{a}_n \neq \dot{a}_j$. □

We fix a $\mathbb{P}$-generic filter G over V and write $M := \mathsf{HS}^G$ for the Cohen model. We write a_n for $\dot{a}_n^G$ and A for $\dot{A}^G$. One can show as in [Jec08, Lemma 5.25 & Lemma 5.26] that $M = V(A)$, the smallest transitive subclass of $V[G]$ that is a model of ZF, contains V and has A as an element.

For a Dedekind finite set A, let $\mathrm{Add}(A,1) := \{p{:}\, F \to 2 \mid F \subseteq A \text{ is finite }\}$ ordered by reverse inclusion. We identify a subset of A with its characteristic function. So $\mathrm{Add}(A,1)$ adds a new subset of A.

Let $f{:}\,\omega \to 2$ be a finite partial function, and let $\dot{q}_f$ denote the following name:

$$\dot{q}_f = \left\{ \langle \dot{a}_n, \check{f(n)} \rangle^\bullet \mid \alpha \in \operatorname{dom} f \right\}^\bullet.$$

Then $\dot{\mathbb{Q}} := \{\dot{q}_f \mid f{:}\,\omega \to 2 \text{ is a finite partial function}\}^\bullet$ is a name for $\mathrm{Add}(\kappa,1)^M$. In particular, if $q \in \mathrm{Add}(A,1)$, then there exists a finite partial function $f{:}\,\omega \to 2$ in V such that $q = \dot{q}_f^G$.

Theorem 3.4. *Suppose that G is an $\mathrm{Add}(A,1)$-generic filter over M. Then M and $M[G]$ have the same sets of ordinals.*

Proof. Let $\dot{X} \in \mathsf{HS}$ be a $\mathbb{P}$-name for an $\mathrm{Add}(A,1)$-name for a set of ordinals. Any $\pi \in \mathscr{G}$ acts on $\mathbb{P} * \mathrm{Add}(\dot{A},1)^\bullet$ via

$$\pi\langle p, \dot{q}_f\rangle = \langle \pi p, \pi\dot{q}_f\rangle = \langle \pi p, \dot{q}_{f\circ\pi}\rangle.$$

We write $\langle p, \dot{q}_f\rangle \Vdash^{\mathsf{HS}} \varphi$ to mean that p forces that $\dot{q}_f \Vdash \varphi$ holds in $V(A)$.

Let $\langle p, \dot{q}_f\rangle$ be a condition which forces that $\dot{X}$ is a name for a set of ordinals. Let E be a support for $\dot{X}$. Then E is a finite subset of ω with $\mathrm{fix}(E) \subseteq \mathrm{sym}(\dot{X})$. We can assume that $\mathrm{supp}(p) = E = \operatorname{dom} f$.

Suppose that $\langle p_0, \dot{q}_{f_0}\rangle$ and $\langle p_1, \dot{q}_{f_1}\rangle$ are two extensions of $\langle p, \dot{q}_f\rangle$. Again, we can assume that $\mathrm{supp}(p_i) = \operatorname{dom} f_i$ for $i < 2$.

We claim that $\dot{X}$ is a name for a set in M. It suffices to show that if $p_1{\upharpoonright}E = p_2{\upharpoonright}E$, then p_0 and p_1 must agree on any statement of the form $\check{\alpha} \in \dot{X}$. This is because there is an automorphism in $\mathrm{fix}(E)$ moving $\mathrm{supp}(p_0) \setminus E$ to be disjoint of $\mathrm{supp}(p_1)$, which means that $\langle \pi p_0, \pi\dot{q}_{f_0}\rangle$ is compatible with $\langle p_1, \dot{q}_{f_1}\rangle$ while $\pi\check{\alpha} = \check{\alpha}$ and $\pi\dot{X} = \dot{X}$. Here we use the fact that $\operatorname{dom} f = E$ and $\operatorname{dom} f_i = E_i$ for $i < 2$. It follows that if $\langle p_i, \dot{q}_{f_i}\rangle \Vdash \check{\alpha} \in \dot{X}$, then $\langle p_i{\upharpoonright}E, \dot{q}_{f_i{\upharpoonright}E}\rangle = \langle p_i{\upharpoonright}E, \dot{q}_f\rangle$ already forces this statement. □

In particular, forcing with $\mathrm{Add}(A,1)$ over M preserves all cardinals and cofinalities.

Remark 3.5. In Cohen's first model, every set is linearly ordered by results of Halpern and Levy. Cohen's second model N witnesses a failure of AC_ω by a sequence $\langle F_n \mid n \in \omega\rangle$ of pairwise disjoint finite sets. In particular, the union A of these sets cannot be linearly ordered. We now force with $\mathrm{Add}(A,1)$ over the model to add a function $g{:}\,A \to 2$. A density argument shows that $\{n \in \omega \mid g[F_n] = \{0\}\}$ is a Cohen real over N. So in contrast to the situation in Cohen's first model, $\mathrm{Add}(A,1)$ adds new reals. A result characterising when this happens for an arbitrary Dedekind finite set A can be found in [KS20, Section 6].

3.2. **Cohen subsets of cardinals.** The forcing $\mathrm{Add}(\kappa,1) = \{p{:}\,\alpha \to 2,\ \alpha < \kappa\}$ ordered by reverse inclusion is not $<\kappa$-closed unless κ is regular. However, studying $\mathrm{Add}(\kappa,1)$ for successors κ tells us much about $<\kappa$-closed forcings, since $\mathrm{Add}(\kappa,1)$ is forcing equivalent (i.e., the two forcings have the same generic extensions) to

$$\mathrm{Add}_*(\kappa,1) := \{(f,g) \mid f \in \mathrm{Add}(\kappa,1),\ g{:}\,\mathrm{dom}(f) \to |\mathrm{dom}(f)| \text{ is bijective}\},$$

ordered by reverse inclusion in the first coordinate. (The second coordinate is not used.) $\mathrm{Add}_*(\lambda^+,1)$ is $<\lambda^+$-closed for any cardinal λ.

Exercise 3.6. Show that $\mathrm{Add}(\kappa,1)$ wellorders $2^{<\kappa}$. In particular, $\mathrm{Add}(\omega_1,1)$ wellorders the reals.

By a $<\lambda$-distributive forcing $\mathbb{P}$, we mean one such that for any sequence $\langle U_i \mid i < \alpha\rangle$ of dense open subsets of $\mathbb{P}$ of length $\alpha < \lambda$, $\bigcap_{i<\alpha} U_i \neq \emptyset$. A λ-distributive forcing does not add element of V^λ. (The converse implication may fail by [KS23].)

Lemma 3.7 (folklore)**.** *For any infinite cardinal λ, DC_λ holds if and only if every $<\lambda^+$-closed forcing is $<\lambda^+$-distributive.*

Proof. Using DC_λ, we can find a sequence $\langle p_i \mid i < \lambda\rangle$ with $p_i \in U_i$ for all $i < \lambda$. Any lower bound p of this sequence is in $\bigcap_{i<\lambda} U_i$. Conversely, if DC_λ fails then there exists a $<\lambda$-closed tree T with no λ-sequences, so T is $<\lambda^+$-closed. Forcing with $(T,\geq)$ adds a new λ-sequence, so T cannot be $<\lambda$-distributive. □

We aim for a similar result for Cohen subsets.

Lemma 3.8. *Suppose that* $\lambda \in \mathrm{Card}$ *and* $\mathbb{P} = \mathrm{Add}(\lambda^+, 1)$. *The following conditions are equivalent:*

(a) $\mathsf{DC}_\lambda(2^\lambda)$.
(b) $\mathbb{P}$ *is* λ*-distributive.*
(c) $\mathbb{P}$ *does not change* V^λ.

Proof. (a) ⇒ (b) ⇒ (c) are as in the previous lemma. (c) ⇒ (a): $\mathbb{P}$ wellorders $(2^\lambda)^{<\lambda}$. Thus the given tree T has a λ-branch in the generic extension. Since $\mathbb{P}$ does not change V^λ, this branch is in V. □

Proposition 3.9 ([IS22, Section 3.5]). *Suppose that* $\lambda \in \mathrm{Card}$ *and* $\mathbb{P} = \mathrm{Add}(\lambda^+, 1)$. *The following conditions are equivalent:*

(a) $\mathsf{DC}_\lambda(2^\lambda)$.
(b) $\mathbb{P}$ *preserves all cardinals* $\alpha \leq \lambda^+$ *and the cofinality of all ordinals* $\alpha \leq \lambda^+$.
(c) $\mathbb{P}$ *preserves* λ^+ *as a cardinal.*
(d) $\mathbb{P}$ *forces that* λ^+ *is regular.*

Proof. (a) ⇒ (b) holds by the previous lemma and (b) ⇒ (c) is clear. (c) ⇒ (d) holds since $\mathbb{P}$ wellorders 2^λ. (d) ⇒ (a): Towards a contradiction, suppose that $\nu \leq \lambda$ is least such that $\mathbb{P}$ adds new elements to V^ν. It suffices to show that $\mathbb{P}$ is ν-distributive. Suppose that $\langle U_i \mid i < \nu \rangle \in V$ is a sequence of dense open subsets of $\mathbb{P}$ and G is a $\mathbb{P}$-generic filter over V. Since $\mathbb{P}$ wellorders $2^{<\lambda^+}$ and does not change $V^{<\nu}$, we can construct a strictly decreasing sequence $\langle p_i \mid i < \nu \rangle$ with $p_i \in U_i \cap G$ in $V[G]$. Since λ^+ is regular in $V[G]$ and $\nu < \lambda^+$, we have $\mu := \sup_{i<\nu} \mathrm{lh}(p_i) < \lambda^+$ and therefore $p := \bigcup_{i<\nu} p_i \in \mathrm{Add}(\lambda^+, 1)$ is the unique condition in G of length μ. In particular, $p \in V$. Hence $p \in \bigcap_{i<\nu} U_i$ as required. □

If $\mathsf{DC}_\nu(2^\nu)$ fails for some $\nu \leq \lambda$, it thus follows that $\mathrm{Add}(\lambda^+, 1)$ collapses λ^+. For example, this holds for all $\lambda \geq \omega_2$ in $L(\mathbb{R})$, assuming there exists no ω_1-sequence of distinct reals in $L(\mathbb{R})$.

Exercise 3.10. Prove that (c) implies (d) in Lemma 3.8.

Problem 3.11. *Which combinations of cardinals* $\leq\lambda$ *can* $\mathrm{Add}(\lambda^+, 1)$ *preserve/collapse?*

3.3. **Adding Cohen subsets over** $L(\mathbb{R})$. Forcing over a model V of ZFC does not change the theory of $L(\mathbb{R})$ if there is a proper class of Woodin cardinals in V by a result of Woodin. Then $L(\mathbb{R})$ satisfies the axiom of determinacy AD and in $L(\mathbb{R})$, AD implies DC by a result of Kechris [Kec84]. A generic extension $L(\mathbb{R})[G]$ of $L(\mathbb{R})$ may be different from $L(\mathbb{R})^{V[G]}$ since the set $\mathbb{R}^V$ of ground model reals might not be in $L(\mathbb{R})^{V[G]}$.

We assume $V = L(\mathbb{R}) \vDash \mathsf{AD}$ and ask which Cohen subsets preserve cardinals and AD.

Cohen reals preserve all cardinals, since the usual argument for ccc forcings works for wellordered forcings. $\mathrm{Add}(\lambda^+, 1)$ collapses λ^+ by Proposition 3.9.

Problem 3.12. *(see Problem 3.11) Which cardinals* $\leq\lambda$ *does* $\mathrm{Add}(\lambda^+, 1)$ *preserve or collapse over* $L(\mathbb{R})$*?*

Exercise 3.13. Suppose that $V = L(\mathbb{R})$ and there exists no ω_1-sequence of distinct reals. Show that $L(\mathbb{R})[G] \vDash \mathrm{cof}(\kappa) \leq |\mathbb{R}|$ for any $\mathrm{Add}(\kappa, 1)$-generic $x \in 2^\kappa$ over $L(\mathbb{R})$ and $\kappa \geq \omega_1$.

Ikegami and Trang observed that a Cohen real destroys AD. This follows from a result of Kunen that $\mathbb{R}^V$ does not have the Baire property adding a Cohen real.

Proposition 3.14 (Chan, Jackson, Goldberg 2021 [CJ21, Fact 3.3]). *Suppose that* $V = L(\mathbb{R}) \vDash \mathsf{AD}$. *Then any well-ordered forcing destroys* AD.

Proof sketch. In $V[G]$, take a perfect tree T with $[T] \subseteq (2^\omega)^V$.

- For each $p \in \mathbb{P}$, let A_p be the set of $x \in 2^\omega$ such that $p \Vdash x \in [T]$.
- Some A_p is uncountable, since a wellordered union of meager sets is meager by Kuratowski-Ulam and the Baire property.

- Take a perfect tree T' with $[T'] \subseteq A_p$. Then $p \Vdash [T'] \subseteq [T]$. Since $\mathbb{P}$ adds new reals, it adds a new element of $[T']$.

□

$\mathrm{Add}(\lambda^+, 1)$ destroys AD, since it forces AC.

However, one can add new subsets of regular cardinals using $\mathrm{Add}(\kappa, 1)^{\mathrm{HOD}}$ while preserving AD.

Definition 3.15. Suppose that M is a transitive model and $A \subseteq \kappa$. A is a *fresh* subset of κ over M if $A \notin M$, but $A \cap \alpha \in$ for all $\alpha < \kappa$.

Theorem 3.16 (Cunningham [Cun23, Theorem 2.4])**.** *Suppose that V is a model of ZFC, $L(\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD}$ and $\kappa > |\mathbb{R}|$ is a regular cardinal. If A is a fresh subset of κ over $L(\mathbb{R})$, then $L(\mathbb{R})[A]$ does not have new sets of reals and hence it is a model of AD.*

Proof. Suppose that $B \in L(\mathbb{R})[A]$ is a set of reals. B is definable over some $L_\alpha(\mathbb{R})[A]$ from a real and an ordinal and we assume that no parameters are needed.

We claim that $B \in L(\mathbb{R})$. If $\alpha < \kappa$, then $L_\alpha(\mathbb{R})[A] = L_\alpha(\mathbb{R})$ since $A \subseteq \kappa$ is fresh and thus $B \in L(\mathbb{R})$.

Suppose that $\alpha \geq \kappa$. Fix an operator H in V for Skolem hulls in $L_\alpha(\mathbb{R})$ and let

$$H_\xi := H^{L_\alpha(\mathbb{R})[A]}(\mathbb{R} \cup \{\xi\})$$

for any $\xi \in \mathrm{Ord}$. Let $M_0 := H_0$. Given M_n, let $\xi_n := M_n \cap \kappa$. Given M_n and ξ_n, let $M_{n+1} := H_{\xi_n}$. Then $M := \bigcup_{n \in \omega} M_n \prec L_\alpha(\mathbb{R})$ and $\xi := M \cap \kappa \in \kappa$, since κ is regular. Let N denote the transitive collapse of M. Since $|M| = |\mathbb{R}| < \kappa$, $N = L_\beta(\mathbb{R})[A \cap \xi]$ for some $\beta < \kappa$. Since B is definable over N, we have $B \in L(\mathbb{R})[A]$. □

The extension $L(\mathbb{R})[A]$ constructed in Theorem 3.16 is in fact a model of DC [Cun23, Theorem 3.3].

The next result shows that one can add fresh subsets to some regular cardinals in $L(\mathbb{R})$.

Theorem 3.17 (Cunningham [Cun23, Theorem 3.3])**.** *Suppose that $L(\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD}$ and $\kappa > (\aleph^*(\mathbb{R})^+)^{L(\mathbb{R})}$ is regular. Let $\mathbb{P} \in \mathrm{HOD}^{L(\mathbb{R})}$ be a forcing such that the $\mathbb{P}$-generic filter is a fresh subset of κ and*

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\kappa \text{ is a regular cardinal”}$$

holds in $\mathrm{HOD}^{L(\mathbb{R})}$. Then AD holds in any $\mathbb{P}$-generic extension of $L(\mathbb{R})$.

Proof sketch. We will assume that κ is a sufficiently large regular cardinal in $L(\mathbb{R})$.

Suppose that $G \times H$ is $\mathbb{P} \times \mathrm{Col}(\omega_1, \mathbb{R})$-generic over $L(\mathbb{R})$, where $\mathrm{Col}(\lambda, X) := \{p : \alpha \to X \mid \alpha < \lambda\}$. Note that G is $\mathbb{P}$-generic over $\mathrm{HOD}^{L(\mathbb{R})}$ as well. Let A be the subset of κ given by G. Then A is a fresh subset of κ over $L(\mathbb{R})$.

We will assume that $\kappa > |\mathbb{R}|^{L(\mathbb{R})[H]}$ is regular. One can check that κ remains regular in $L(\mathbb{R})[H]$ [Cun23, Theorem 2.7].

It suffices to show that κ is regular in $L(\mathbb{R})[G \times H]$. Since this is a model of choice, we can apply Theorem 3.16 to A in this model and will thus obtain $L(\mathbb{R})[G] \models \mathsf{AD}$.

Let $M := \mathrm{HOD}^{L(\mathbb{R})}$. Since $\mathrm{Col}(\omega_1, \mathbb{R})$ is homogeneous, we have $M = \mathrm{HOD}^{L(\mathbb{R})[H]}$. Let $\mathbb{Q}$ denote Vopenka's forcing in M for subsets of ω_1 [Jec03]. Then any subset of ω_1 in $L(\mathbb{R})[H]$ is $\mathbb{Q}$-generic over M. In particular, this holds for the subset of ω_1 given by H. Since this set codes all reals, we have that $L(\mathbb{R})[H] = M[H]$ is a $\mathbb{Q}$-generic extension of M and

$$M[G][H] = M[H][G] = L(\mathbb{R})[G \times H]$$

By the assumption of the theorem, κ remains regular in $M[G]$. We now assume $\kappa > |\mathbb{Q}|^M$ so that κ remains regular in $M[G][H]$ as required.

To see that the lower bound on κ suffices, one needs to check that $\omega_2^{\mathrm{HOD}[H]} = \aleph^*(\mathbb{R})^{L(\mathbb{R})}$ and $|\mathbb{Q}|^M \leq \omega_3^{\mathrm{HOD}[H]} = (\aleph^*(\mathbb{R})^+)^{L(\mathbb{R})}$ [Cun23, Theorem 2.7]. □

Remark 3.18. Ikegami and Trang proved a stronger version of Theorem 3.17 for all $\kappa \geq \aleph^*(\mathbb{R})^{L(\mathbb{R})}$ [IT23, Theorem 5.1]. Chan and Jackson proved that $\aleph^*(\mathbb{R})$ is least, since any forcing over $L(\mathbb{R})$ that is a surjective image of $\mathbb{R}$ destroys AD [CJ21, Theorem 5.6].

4. Chain conditions

4.1. Variants of the ccc.

Definition 4.1 (Karagila, Schweber [KS22])**.**
- ccc_1: Every maximal antichain in $\mathbb{P}$ is countable.
- ccc_2: Every antichain in $\mathbb{P}$ is countable.
- ccc_3: Every predense subset of $\mathbb{P}$ contains a countable predense subset.

Moreover, ccc_i^* means ccc_i restricted to wellordered antichains, or predense subsets, of $\mathbb{P}$.

These notions are equivalent for well-orderable forcings. Karagila and Schweber [KS22] showed that the implications

$$\text{ccc}_3 \Rightarrow \text{ccc}_2 \Rightarrow \text{ccc}_1$$

are provable in ZF, but none of these implications can be reversed in $\mathsf{ZF} + \mathsf{DC}$.

Exercise 4.2. There exists a ccc_2^* forcing which collapses ω_1 if there is no ω_1-sequence of distinct reals.

The following theorem of Bukovsky gives us a new variant of the ccc that we call Bukovsky's condition.

Theorem 4.3 (Bukovsky)**.** *Suppose that $V \subseteq W$ are models of* ZFC*. Then W is a generic extension of V by a ccc forcing if and only if for every $x \in V$ and $f : x \to V$ in W, there exists a function $g : x \to V$ such that*

(1) $V \vDash |g(u)| < \omega_1$ for all $u \in x$, and
(2) $W \vDash f(u) \in g(u)$ for all $u \in x$.

Their theorem holds for the κ-cc for other regular κ as well.

Lemma 4.4 (Karagila, Schweber [KS22])**.** ccc_3 *implies Bukovsky's condition.*

Problem 4.5 (Karagila, Schweber [KS22])**.** *Does Bukovsky's condition imply* ccc_3*?*

Proposition 4.6 (Karagila, Schweber [KS22])**.** *If $\mathbb{P}$ satisfies Bukovský's condition, then $\mathbb{P}$ preserves any cardinal $\kappa > \omega_1$. If ω_1 is regular, then it is not collapsed.*

Proof. Suppose that $\kappa < \lambda$ are cardinals and $f : \kappa \to \lambda$ is a surjective function in $V[G]$. Pick some $F : \kappa \to [\lambda]^{<\omega_1}$ such that $f(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$. Since f is surjective, $\bigcup_{\alpha<\kappa} F(\alpha) = \lambda$. But $\bigcup_{\alpha<\kappa} F(\alpha)$ has size at most $\kappa \cdot \omega_1 = \kappa$. If ω_1 is regular, $\kappa = \omega$ and $\lambda = \omega_1$, then $\bigcup_{n<\omega} F(n)$ is countable. □

Problem 4.7 (Karagila, Schweber [KS22])**.** *Is it consistent that a* ccc_3 *forcing collapses ω_1?*

4.2. σ-linked forcings.

Definition 4.8. A forcing $\mathbb{P}$ is κ-linked if there exists a (linking) function $f : \mathbb{P} \to \kappa$ such that for all $p, q \in \mathbb{P}$,

$$f(p) = f(q) \Rightarrow p \parallel q.$$

ω-linked is also called σ-linked.

Any κ-linked forcing is split into κ many pieces, each one consisting of pairwise compatible conditions.

Example 4.9. Hechler forcing is defined as $\mathbb{H} := \{(s, f) \mid s \in \omega^{<\omega},\ f \in \omega^\omega,\ s \subseteq f\}$, where $(t, g) \leq (s, f)$ if $s \subseteq t$ and $f(n) \leq g(n)$ for all $n \in \omega$. $\mathbb{H}$ is σ-linked.

Every σ-linked forcing satisfies ccc_2.

Problem 4.10. *Does every σ-linked forcing satisfy* ccc_3*?*

The definition of κ-linked could say

$$p \perp q \Rightarrow f(p) \neq f(q).$$

and we can equip Ord with the discrete partial order $=$. This suggests a generalisation of κ-linked relative to a forcing $\mathbb{Q}$:

Definition 4.11. $\mathbb{P}$ is $\mathbb{Q}$-linked if there exists a $\perp$-homomorphism $f\colon\mathbb{P}\to\mathbb{Q}$, i.e., such that for all $p,q\in\mathbb{P}$

$$p\perp q\Rightarrow f(p)\perp f(q).$$

Note that in ZFC, if $\mathbb{P}$ is $\mathbb{Q}$-linked and $\mathbb{Q}$ is ccc, then $\mathbb{P}$ is ccc.

Exercise 4.12. Well-ordered c.c.c. forcings preserve cardinals. (To see this, work in HOD with the relevant parameters.)

$\mathbb{C}:=\{p\mid p{:}\,n\to 2, n\in\omega\}$ denotes Cohen forcing and $\mathbb{C}^\kappa$ the finite support product of κ many copies. They are well-ordered.

This goes further:

Lemma 4.13. *Suppose that $\mathbb{P}$ is $\mathbb{Q}$-linked and $\mathbb{Q}$ is well-ordered and c.c.c. Then $\mathbb{P}$ preserves all cardinals.*

Proof sketch. Suppose that $1_\mathbb{P}\Vdash \dot f{:}\,\omega\to\check\omega_1$ is surjective. Let $g{:}\,\mathbb{P}\to\mathbb{Q}$ be a $\perp$-homomorphism. Define $q\Vdash^*\varphi\Leftrightarrow\exists p\ \ f(p)=q\wedge p\Vdash\varphi$. If $q\Vdash^*\varphi$, $q'\Vdash^*\psi$ and φ, ψ are contradictory, then $q\perp q'$ as

$$p\Vdash\varphi\wedge p'\Vdash\psi\Rightarrow p\perp p'\Rightarrow f(p)\perp f(p').$$

Let A_n be a maximal antichain of $q\in\mathbb{Q}$ with $q\Vdash^*$ "$\dot f(n)=\alpha$" This can be done in $M:=\mathrm{HOD}_{\{\mathbb{P},\mathbb{Q},\dot f\}}$, since $\mathbb{Q}\subseteq M$. In M, ω_1^V is regular, $\bigcup_{n\in\omega}A_n$ is countable and $\omega_1^V\leq^*\bigcup_{n\in\omega}A_n$. □

Exercise 4.14. Let $\mathbb{P}_\alpha$ denote α with the discrete partial order. Then $\prod_{\alpha<\omega_1}\mathbb{P}_\alpha$ collapses ω_1.

We therefore need a uniformity requirement on an iteration.

A product or iteration of σ-linked forcings is called uniform if it comes with a sequence of names for linking functions.

Theorem 4.15. *Any uniform finite support iteration of σ-linked forcings of length κ is $\mathbb{C}^\kappa$-linked.*

Hence cardinals are preserved.

Problem 4.16. *Do Cohen and Hechler models over V have different theories?*

- *A Cohen model is a $\mathbb{C}^\kappa$-generic extension for some $\kappa\geq\omega_2$.*
- *A Hechler model is obtained by a finite support iteration of $\mathbb{H}$ of some length $\kappa\geq\omega_2$.*

Proposition 4.17. *Any uniform finite support iteration of σ-linked forcings of length κ is $\mathbb{C}^\kappa$-linked.*

Proof sketch. Let $\langle\mathbb{P}_\alpha,\dot{\mathbb{P}}_\alpha,\dot f_\alpha\mid\alpha<\kappa\rangle$ denote such an iteration, where $\dot f_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-linking function for $\dot{\mathbb{P}}_\alpha$.

One can show that the set $\tilde{\mathbb{P}}$ of all $p\in\mathbb{P}_\kappa$ such that for all $\alpha\in\mathrm{supp}(p)$, $p{\restriction}\alpha$ decides $\dot f_\alpha(p(\alpha))$, is dense [IS22, Lemma 4.16].

We can use the values of these functions to read off a $\perp$-homomorphism from $\tilde{\mathbb{P}}$ to the set $\mathrm{Fun}_{<\omega}(\kappa,\omega)$ of finite partial functions $p{:}\,\kappa\to\omega$. $\mathrm{Fun}_{<\omega}(\kappa,\omega)$ can be densely embedded into $\mathbb{C}^\kappa$. □

4.3. **Narrow forcings.** The following is just the ccc_2^* for $\mathbb{B}(\mathbb{P})$.

Definition 4.18. $\mathbb{P}$ is called $(\omega,1)$-narrow if all partial $\|$-homomorphisms $f{:}\,\mathbb{P}\rightharpoonup\mathrm{Ord}$ have countable range.

- A partial $\|$-homomorphism f corresponds to a function on the set D all $p\in\mathbb{P}$ deciding a statement, for instance $p\Vdash\dot g(n)=\alpha_p$. f sends $p\in D$ to α_p.
- A partial $\|$-homomorphism f can be thought of a generalised antichain consisting of "blocks" $f^{-1}(\alpha)$. Different blocks are incompatible.
- In a complete Boolean algebra, a partial $\|$-homomorphism corresponds to an antichain, since subsets A and B of $\mathbb{P}$ are elementwise incompatible if and only if $\sup(A)$ is incompatible with $\sup(B)$.

However, when trying to prove cardinal preservation via a function $\dot{f}:\omega \to \omega_1$, an ω-sequence of such homomorphisms appears. This is captured by the next uniform version of ccc_2^* for many homomorphisms.

We will see that all σ-linked forcings are narrow and later that random algebras are narrow.

Definition 4.19. Suppose that $\mathbb{P}$ is a forcing and θ, ν are ordinals, where θ is infinite.

1. $\mathbb{P}$ is called (θ,ν)-*narrow* if for any ordinal $\mu \leq \nu$ and any sequence $\vec{f} = \langle f_i \mid i < \mu\rangle$ of partial $\|$-homomorphisms $f_i:\mathbb{P} \rightharpoonup \text{Ord}$,
$$|\bigcup_{i<\mu} \text{ran}(f_i)| \leq |\max(\theta,\mu)|.$$
2. $\mathbb{P}$ is called θ-*narrow* if it is (θ,ν)-narrow for all $\nu \in \text{Ord}$.

We further call $\mathbb{P}$ *uniformly* θ-*narrow* if there exists a function G that sends each partial $\|$-homomorphism $f:\mathbb{P} \rightharpoonup \text{Ord}$. We can assume that $\text{ran}(f)$ is an ordinal. to an injective function $G(f):\text{ran}(f) \to \theta$.

We further omit ω, so narrow means ω-narrow etc.

Note that (θ,θ)-narrow already implies (θ,ν)-narrow for all ν, since $(\theta,1)$-narrow implies (θ,μ)-narrow for all $\mu \geq \theta^+$ by cardinal arithmetic. Moreover, any uniformly θ-narrow forcing is θ-narrow.

Any wellordered θ^+-c.c. forcing $\mathbb{P}$ is θ-narrow. This can be seen by working in $\text{HOD}_{\mathbb{P},\vec{f}}$ for any $\vec{f}$ as above, since $\mathbb{P} \cap \text{HOD}_{\mathbb{P},\vec{f}}$ is ν-c.c. in $\text{HOD}_{\mathbb{P},\vec{f}}$ for some $\nu < \theta^+$ if θ^+ is singular in $\text{HOD}_{\mathbb{P},\vec{f}}$.

Note that if θ^+ is regular, then $(\theta,1)$-narrow implies θ-narrow. Moreover, if there exists a sequence of injective functions from all $\alpha < \theta^+$ into θ, then $(\theta,1)$-narrow implies uniformly θ-narrow.

We do not know if every $(\theta,1)$-narrow forcing is θ-narrow and whether every θ-narrow forcing is uniformly θ-narrow. Moreover, we do not know if $(\theta,1)$-narrow forcings preserve θ^+ for all $\theta \in \text{Card}$. It is true for ω_1 by an argument with Schilhan and Karagila, but this does not generalise. θ-narrow forcings preserve all cardinals $>\theta$ by the next lemma.

Lemma 4.20.

1. *Every $(\theta,1)$-narrow forcing $\mathbb{P}$ preserves all cardinals and cofinalities $\geq\theta^{++}$.*
2. *Every θ-narrow forcing $\mathbb{P}$ preserves all cardinals and cofinalities $\geq\theta^+$.*

Proof. 1. We first show that $\mathbb{P}$ preserves any cardinal $\lambda \geq \theta^{++}$. Suppose that $\mu < \lambda$ is a cardinal, $\dot{f}$ is a $\mathbb{P}$-name, and $p \Vdash_{\mathbb{P}} \dot{f}:\mu \to \lambda$ for some $p \in \mathbb{P}$. For each $\alpha < \mu$, let D_α denote the set of all $q \leq p$ in $\mathbb{P}$ that decide $\dot{f}(\alpha)$. Define $f_\alpha:D_\alpha \to \lambda$ by sending each q to the unique $\beta < \lambda$ with $q \Vdash \dot{f}(\alpha) = \beta$. Note that each f_α is a partial $\|$-homomorphism on $\mathbb{P}$. Since $\mathbb{P}$ is $(\theta,1)$-narrow, $\text{otp}(\text{ran}(f_\alpha)) < \theta^+$ for each $\alpha < \mu$. Hence $|\bigcup_{\alpha<\mu} \text{ran}(f_\alpha)| \leq |\max(\theta^+,\mu)| < \lambda$. Hence p forces that $\dot{f}$ is not surjective.

A similar argument works for cofinalities. Suppose that λ is a cardinal with $\text{cof}(\lambda) \geq \theta^{++}$. Suppose that $\mu < \text{cof}(\lambda)$ is a cardinal, $\dot{f}$ is a $\mathbb{P}$-name, and $p \Vdash_{\mathbb{P}} \dot{f}:\mu \to \lambda$ for some $p \in \mathbb{P}$. With the same notation as above, $|\bigcup_{\alpha<\mu} \text{ran}(f_\alpha)| \leq |\max(\theta^+,\mu)| < \text{cof}(\lambda)$, so p forces that $\dot{f}$ is not cofinal.

2. We first show that $\mathbb{P}$ preserves θ^+. Suppose that $\mu < \theta^+$ is a cardinal, $\dot{f}$ is a $\mathbb{P}$-name, and $p \Vdash_{\mathbb{P}} \dot{f}:\mu \to \theta^+$ for some $p \in \mathbb{P}$. For each $\alpha < \mu$, let D_α denote the set of all $q \leq p$ in $\mathbb{P}$ that decide $\dot{f}(\alpha)$. Define $f_\alpha:D_\alpha \to \theta^+$ by sending each q to the unique $\beta < \theta^+$ with $q \Vdash \dot{f}(\alpha) = \beta$. Note that each f_α is a partial $\|$-homomorphism on $\mathbb{P}$. Since $\mathbb{P}$ is θ-narrow, we have $|\bigcup_{\alpha<\mu} \text{ran}(f_\alpha)| \leq |\max(\theta,\mu)| < \theta^+$. Hence p forces that $\dot{f}$ is not surjective.

A similar argument works for cofinality θ^+. Suppose that λ is a cardinal with $\text{cof}(\lambda) = \theta^+$. Suppose that $\mu < \text{cof}(\lambda)$ is a cardinal, $\dot{f}$ is a $\mathbb{P}$-name, and $p \Vdash_{\mathbb{P}} \dot{f}:\mu \to \lambda$ for some $p \in \mathbb{P}$. With the same notation as above, $|\bigcup_{\alpha<\mu} \text{ran}(f_\alpha)| \leq |\max(\theta,\mu)| < \text{cof}(\lambda)$, so p forces that $\dot{f}$ is not cofinal. □

Lemma 4.21. *Suppose that θ, ν are cardinals, where θ is infinite, and $f:\mathbb{P} \to \mathbb{Q}$ is a $\perp$-homomorphism.*

1. $\mathbb{Q}$ *is* (θ,ν)*-narrow, then* $\mathbb{P}$ *is* (θ,ν)*-narrow.*
2. $\mathbb{Q}$ *is uniformly* θ*-narrow, then* $\mathbb{P}$ *is uniformly* θ*-narrow.*

Proof. 1. Suppose that $\vec{f} = \langle f_i \mid i < \mu\rangle$ is a sequence of partial $\|$-homomorphisms $f_i\colon \mathbb{P} \rightharpoonup \mathrm{Ord}$. Let $D := \mathrm{ran}(f)$ and define $g_i\colon D \to \mathrm{Ord}$ as follows. Note that for all $p, r \in \mathbb{P}$ with $f(p) = f(r)$, we have $f_i(p) = f_i(r)$, since f is a $\perp$-homomorphism and f_i is a $\|$-homomorphism. For $f(p) = q \in D$, we can thus define $g_i(q) = f_i(p)$. We claim that each g_i is a partial $\|$-homomorphism. Suppose that $q, s \in D$ with $f(p) = q$, $f(r) = s$ and $q \parallel s$. Since f is a $\perp$-homomorphism, $p \parallel r$. Since f_i is a $\|$-homomorphism, $g_i(q) = f_i(p) \parallel f_i(r) = g_i(s)$ as desired. Since $\mathrm{ran}(f_i) = \mathrm{ran}(g_i)$ for all $i < \mu$ and $\mathbb{Q}$ is (θ,ν)-narrow, the statement of the lemma follows.

2. Suppose G witnesses that $\mathbb{Q}$ is uniformly θ-narrow. The proof of 1. defines a function H from G that witnesses $\mathbb{P}$ is uniformly θ-narrow by mapping a partial $\|$-homomorphism $f\colon \mathbb{P} \rightharpoonup \mathrm{Ord}$ to a partial $\|$-homomorphisms g on $\mathbb{Q} \rightharpoonup \mathrm{Ord}$ with $\mathrm{ran}(f) = \mathrm{ran}(g)$. □

4.4. **Iterations.** Suppose that θ is an infinite ordinal. A *uniform finite support iteration* of uniformly θ-narrow forcings is a sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\beta, \dot{g}_\beta \mid \alpha \le \delta, \beta < \delta\rangle$ such that $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\beta \mid \alpha \le \delta, \beta < \delta\rangle$ is a finite support iteration and for each $\alpha < \delta$, $\mathbb{1}_{\mathbb{P}_\alpha}$ forces that $\dot{\mathbb{P}}_\alpha$ is uniformly θ-narrow as witnessed by $\dot{G}_\alpha$.

Theorem 4.22. *Suppose that* θ *is an infinite ordinal. Any uniform finite support iteration of uniformly* θ*-narrow forcings is again uniformly* θ*-narrow.*

Proof. We can assume $\theta \in \mathrm{Card}$. Let $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\beta, \dot{g}_\beta \mid \alpha \le \delta, \beta < \delta\rangle$ denote the iteration. We construct a sequence $\langle G_\gamma \mid \gamma \le \delta\rangle$ of functions by recursion on $\gamma \le \delta$ from $\vec{\mathbb{P}}$ and θ, where G_γ witnesses that $\mathbb{P}_\gamma$ is uniformly θ-narrow.

We can assume that each G_γ is a set function by taking $\mathrm{ran}(f)$ to be an ordinal for any argument f of G_γ.

Case. γ is a successor.

Suppose that $\gamma = \beta + 1$ and G_β has been constructed. Let $f\colon \mathbb{P}_\beta * \dot{\mathbb{P}}_\beta \rightharpoonup \mathrm{Ord}$ be a partial $\|$-homomorphism and

$$\dot{f} := \{((\dot{q}, \check{\alpha})^\bullet, p) \mid f(p, \dot{q}) = \alpha\}.$$

Claim. $\mathbb{1}_{\mathbb{P}_\beta}$ forces that $\dot{f}$ is a partial $\|$-homomorphism on $\dot{\mathbb{P}}_\beta$.

Proof. Let G be a $\mathbb{P}_\beta$-generic filter over V and work in $V[G]$. Suppose that $q_0, q_1 \in \dot{\mathbb{P}}_\beta^G$ and $\dot{f}^G(q_i) = \alpha_i$ for $i < 2$. By the definition of $\dot{f}$, there exist $\dot{q}_i$ with $\dot{q}_i^G = q_i$ and $p_i \in G$ with $((\dot{q}_i, \check{\alpha}_i), p_i)^\bullet \in \dot{f}$ for $i < 2$.

Suppose that $\alpha_0 \ne \alpha_1$. We claim that $q_0 \perp q_1$. Otherwise some $p \in G$ forces $\dot{q}_0 \| \dot{q}_1$ and we may assume $p \le p_0, p_1$. Then $(p_0, \dot{q}_0)$ and $(p_1, \dot{q}_1)$ would be compatible and $f(p_0, \dot{q}_0) = \alpha_0 \ne \alpha_1 = f(p_1, \dot{q}_1)$, while f is a $\|$-homomorphism. □

Therefore, $\mathbb{1} \Vdash_{\mathbb{P}_\beta} \dot{g}_\beta(\dot{f})\colon \mathrm{ran}(\dot{f}) \to \theta$ is injective. We can read off a $\mathbb{P}_\beta$-name $\dot{h}$ for a function extending $\dot{g}_\beta(\dot{f})^{-1}$. Then $\mathbb{1} \Vdash_{\mathbb{P}_\beta} \dot{h}\colon \theta \to \mathrm{ran}(\dot{f})$ is surjective.

For each $\alpha < \theta$, let D_α denote the set of all $p \in \mathbb{P}_\beta$ that decide $\dot{h}(\alpha)$ and let $h_\alpha\colon D_\alpha \to \mathrm{Ord}$ by the $\|$-homomorphism defined by letting $h_\alpha(p)$ be the unique δ such that $p \Vdash \dot{h}(\alpha) = \delta$.

Since G_β witnesses that $\mathbb{P}_\beta$ is uniformly θ-narrow, the sequence $\langle G_\beta(h_\alpha) \mid \alpha < \theta\rangle$ consists of injective functions $G_\beta(h_\alpha)\colon \mathrm{ran}(h_\alpha) \to \theta$. We thus obtain an injective function $i\colon \bigcup_{\alpha<\theta} \mathrm{ran}(h_\alpha) \to \theta$.

Since $\mathbb{1}_{\mathbb{P}} \Vdash \mathrm{ran}(\dot{f}) \subseteq \bigcup_{\alpha<\theta} \mathrm{ran}(h_\alpha)$, we have $\mathrm{ran}(f) \subseteq \bigcup_{\alpha<\theta} \mathrm{ran}(h_\alpha)$ by the definition of $\dot{f}$. Thus $i{\restriction}\mathrm{ran}(f) \to \theta$ is injective. Let $G_\gamma(f) := i{\restriction}\mathrm{ran}(f)$.

Case. γ is a limit.

Suppose that $f\colon \mathbb{P}_\gamma \rightharpoonup \mathrm{Ord}$ is a partial $\|$-homomorphism. It suffices to show $\mathrm{HOD}_{\vec{\mathbb{P}},f} \vDash \mathrm{ran}(f) \le \theta$, since we then take the least injective function $G_\gamma(f)\colon \mathrm{ran}(f) \to \theta$ in $\mathrm{HOD}_{\vec{\mathbb{P}},f}$ in its canonical wellorder.

To see this, let $s_\alpha \in [\gamma]^{<\omega}$ for each $\alpha \in \mathrm{ran}(f)$ be least with respect to a fixed definable wellorder of $[\mathrm{Ord}]^{<\omega}$ such that there exists some $p \in \mathbb{P}_\gamma$ with support s_α and $f(p) = \alpha$. Let

$\vec{s} = \langle s_\alpha \mid \alpha \in \operatorname{ran}(f)\rangle$. By restricting f, we can assume that for all $\alpha \in \operatorname{ran}(f)$, all $p \in \mathbb{P}_\gamma$ with $f(p) = \alpha$ have support s_α.

Suppose that $\mathrm{HOD}_{\vec{\mathbb{P}},f} \vDash \operatorname{ran}(f) > \theta$ towards a contradiction. We can assume $\mathrm{HOD}_{\vec{\mathbb{P}},f} \vDash \operatorname{ran}(f) = \theta^+$ by restricting f. Work in $\mathrm{HOD}_{\vec{\mathbb{P}},f}$. By the Δ-system lemma, we obtain a subfamily of $\vec{s}$ that forms a Δ-system with a root r. We will assume that $\vec{s}$ is already a Δ-system. Fix some $\gamma' < \gamma$ such that $\alpha + 1 < \gamma_0$ for all $\alpha \in r$ and let $D := \{p{\restriction}\gamma' \mid p \in \operatorname{dom}(f)\}$ be the projection of $\operatorname{dom}(f)$ to $\mathbb{P}_{\gamma'}$. The function $g\colon D \to \mathrm{Ord}$ defined by $g(p) := \alpha$ if

$$\exists q \in \operatorname{dom}(f) \quad (q{\restriction}\gamma' = p \wedge f(q) = \alpha)$$

is well-defined by the next claim.

Claim. If $u, v \in \operatorname{dom}(f)$ with $u{\restriction}\gamma' = v{\restriction}\gamma' = p \in D$, then $f(u) = f(v)$.

Proof. Suppose that $f(u) = \alpha$ and $f(v) = \beta$. Then $\operatorname{supp}(u) = s_\alpha$ and $\operatorname{supp}(v) = s_\beta$. Since $\vec{s}$ is a Δ-system with root r, $s_\alpha \cap s_\beta \subseteq \gamma'$ and $u \parallel v$. Since f is a partial $\parallel$-homomorphism, $f(u) \neq f(v)$. □

Claim. $g\colon \mathbb{P}_\beta \rightharpoonup \mathrm{Ord}$ is a $\parallel$-homomorphism.

Proof. Suppose that $p, q \in D$ with $g(p) = \alpha$ and $g(q) = \beta$. By the definition of g, there exist $u, v \in \operatorname{dom}(f)$ with $u{\restriction}\gamma' = p$, $f(u) = \alpha$, $v{\restriction}\beta = q$ and $f(v) = \beta$.

If $\alpha \neq \beta$, then $u \perp v$, since f is a partial $\parallel$-homomorphism. Since $\operatorname{dom}(u) \cap \operatorname{dom}(v) = s_\alpha \cap s_\beta \subseteq \gamma'$, $p = t{\restriction}\gamma'$ and $q = u{\restriction}\gamma'$ are incompatible. □

Claim. $\operatorname{ran}(f) = \operatorname{ran}(g)$.

Proof. Suppose that $\alpha \in \operatorname{ran}(f)$ and $f(p) = \alpha$. Then $p{\restriction}\gamma' \in D$ and $g(p{\restriction}\gamma') = \alpha$ by the definition of g. □

The inductive hypothesis for γ' yields an injective function $G_{\gamma'}(g)\colon \operatorname{ran}(g) \to \theta$. Since $G_{\gamma'}, g \in \mathrm{HOD}_{\vec{\mathbb{P}},f}$, we have $\mathrm{HOD}_{\vec{\mathbb{P}},f} \vDash \operatorname{ran}(f) = \operatorname{ran}(g) \leq \theta$, contradicting the assumption. □

4.5. **A** ccc_2 **forcing collapsing** ω_1**.** The following result uses a standard technique for symmetric models that appeared in work of Hodges [Hod74].

Let $\mathcal{L}$ be a first-order language and let M be an $\mathcal{L}$-structure. Given a group $\mathscr{G} \subseteq \operatorname{Aut}(M)$ and an ideal $\mathscr{I}$ of subsets of M, we say that a subgroup of $\mathscr{G}$ is *large* if it contains $\operatorname{fix}(A) = \{\pi \in \mathscr{G} \mid \pi{\restriction}A = \mathrm{id}\}$ for some $A \in \mathscr{I}$. Given $\mathcal{L}, M, \mathscr{G}$ and $\mathscr{I}$, we call a subset X of M *stable* if there exists a large subgroup $\mathscr{H}$ of $\mathscr{G}$ such that $\pi[X] = X$ for all $\pi \in \mathscr{H}$.

Theorem 4.23 (Karagila, Schweber [KS22, Theorem 3.2])**.** *In a model of choice, let $\mathcal{L}$, M, $\mathscr{G}$ and $\mathscr{I}$ be as above. There is a symmetric extension of the universe in which there exists an isomorphic (with isomorphism existing in some further extension) copy N of M such that every subset of N^k in the symmetric extension is a stable isomorphic copy of a subset of M^k. In addition, we can require:*

- $\mathsf{DC}_{<\kappa}$ *holds in the extension, if $\mathscr{I}$ is $<\kappa$-complete.*
- *The extension has no new λ-sequences for any prescribed cardinal λ.*

Theorem 4.24 (Karagila, Schweber [KS22, Theorem 6.4])**.** *It is consistent with $\mathsf{ZF} + \mathsf{DC}$ that there exists a ccc_2 forcing which collapses ω_1.*

Proof. We first construct a symmetric model over a model of ZFC. Let $\mathbb{P}$ denote $\mathrm{Add}(\omega, \omega_1)$ without $\mathbb{1}$. $\mathbb{P}$ is productively c.c.c.

Consider the lottery sum $\mathbb{P}_\infty := \bigoplus_{\langle n,\alpha\rangle \in \omega\times\omega_1} \mathbb{P}_{n,\alpha}$, where each $\mathbb{P}_{n,\alpha}$ is an isomorphic copy of $\mathbb{P}$. We assume the $\mathbb{P}_{n,\alpha}$ are disjoint, so we can understand $\mathbb{P}_\infty$ as their disjoint union. Let $\mathscr{G}$ by the group acts on each $\mathbb{P}_{n,\alpha}$ individually for countably many pairs $\langle n, \alpha\rangle$ at the same time. Let $\mathscr{I}$ be the ideal of countable subsets of $\mathbb{P}_\infty$.

By Theorem 4.24, we get a symmetric extension M of V and, working in M, an isomorphic copy of $\mathbb{P}_\infty$, such that M is a model of DC and ω_1 remains uncountable in M. We will again denote the copy by $\mathbb{P}_\infty$ and its summands by $\mathbb{P}_{n,\alpha}$. We now work in M. For any subset A of $\mathbb{P}_\infty^k$, there is a countable $\alpha < \omega_1$ such that if $\alpha \leq \beta$ and $p(i) \in \mathbb{P}_{n,\beta}$ for any $p \in A^k$, $i < k$ and

$n \in \omega$, then any condition q obtained by replacing $p(i)$ by an arbitrary condition in $\mathbb{P}_{n,\alpha}$ is in A.

Working in this symmetric extension consider the partial order $\mathbb{Q}$ given by pairs $\langle t, \vec{b}\rangle$ such that:

1. $t \in \omega_1^{<\omega}$ and $\mathrm{dom}(t) = n$.
2. $\vec{b} = \langle b_0, \dots, b_{n-1}\rangle$ and $b_i \in \mathbb{P}_{i,t(i)}$.

Let $\langle t, \vec{b}\rangle \leq \langle t', \vec{b}'\rangle$ if the following conditions hold:

1. $t' \subseteq t$.
2. For all $i \in \mathrm{dom}(t')$, $b_i \leq_{n,\alpha} b'_i$.

This is a two-step iteration that first adds a surjection $f\colon \omega \to \omega_1$ using finite conditions and then forces with the product $\prod_{\langle n,\alpha\rangle} \mathbb{P}_{n,\alpha}$. Thus, forcing with $\mathbb{Q}$ collapses ω_1.

It remains to prove that every antichain in $\mathbb{Q}$ is countable. Let π denote the projection of $\mathbb{Q}$ to $\omega_1^{<\omega}$ and $\pi_{n,\alpha}$ the projection to $\mathbb{P}_{n,\alpha}$. Suppose that D is an uncountable subset of $\mathbb{P}_\infty$.

It suffices to show that $\pi^{-1}(t) \cap D$ is uncountable for some $t \in \omega_1^{<\omega}$. Then $\pi^{-1}(t) \cap D$ contains two incompatible conditions, since it is a subset of $\{t\} \times \prod_{i\in\mathrm{dom}(t)} \mathbb{P}_{i,t(i)}$ and $\mathbb{P} = \mathrm{Add}(\omega, \omega_1)$ is productively ccc.

Case. $\pi(D)$ is countable. Then by DC, there exists some $t \in \omega_1^{<\omega}$ such that $\pi^{-1}(t) \cap D$ is uncountable.

Case. $\pi(D)$ is uncountable. We can assume that for some $k \in \omega$, $\mathrm{dom}(t) = k$ for all $t \in \pi(D)$ by shrinking D. We can then identify D with a subset of $\mathbb{P}^k_\infty$. Pick $\alpha < \omega_1$ as above. Since $\pi(D)$ is uncountable, there exists some $t \in \pi(D)$ with $t(i) \geq \alpha$ for some $i < k$. Then $\pi^{-1}(t) \cap D$ is uncountable. □

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