Forcing over choiceless models

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We aim to develop a theory of forcing over arbitrary choiceless models of set theory.

This talk is based on joint work with Daisuke Ikegami (Guangzhou).

- Daisuke Ikegami, Philipp Schlicht:
  Forcing over choiceless models and generic absoluteness, 28 pages
  submitted

Introduction

Mathematics without choice

Set theory without the axiom of choice allows us to do a lot of basic mathematics.

- Many theorems in analysis, for example the intermediate value theorem
- Algebra of countable groups and fields
- Theorems studied in second order arithmetic and reverse mathematics
- Transfinite induction and recursion

Mathematics without choice

However, many things can go wrong:

- Basic measure theory
- Much of functional analysis
- Existence of maximal ideals in rings
- Existence of nontrivial ultrafilters
- Existence of uncountable regular cardinals

Mathematics without choice

Solovay's model is obtained by forcing with

$$\mathrm{Col}(\omega, {<}\kappa) = \prod_{\alpha<\kappa} \mathrm{Col}(\omega, \alpha),$$

where $\mathrm{Col}(\omega, \alpha) = \{p \colon n \to \alpha\}$ and κ is an inaccessible cardinal.

- Solovay's model is the $L(\mathbb{R})$ of this generic extension.

Solovay (1970) showed that in this model, every set of reals is Lebesgue measurable.

Forcing with choice

The following models coincide:

- Solovay's model over a universe with larger cardinals
- $L(\mathbb{R})$ in a universe with larger cardinals.

These models are very well understood and satisfy stronger versions of Solovay's theorem concerning the determinacy of infinite games.

Forcing with choice

A forcing is a set $\mathbb{P}$ with a separative partial order $\leq$. $\mathbb{P}$ can be embedded into a complete Boolean algebra $\mathbb{B} := \mathbb{B}(\mathbb{P})$.

A $\mathbb{B}$-valued model of ZFC is constructed as

- $V_0^{\mathbb{B}} = \emptyset$
- $V_{\alpha+1}^{\mathbb{B}} = \{f \colon V_\alpha^{\mathbb{B}} \to \mathbb{B}\}$
- $V_\lambda^{\mathbb{B}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbb{B}}$ for limits λ

with union $V^{\mathbb{B}}$. If G is a filter on $\mathbb{B}$, $V[G] := V^{\mathbb{B}}/G$ is a model of ZFC that contains a copy of V via 0-1-valued functions. Suppose G is generic.

Forcing with choice

The countable (anti)chain condition for $\mathbb{P}$ states that there are no uncountable antichains in $\mathbb{P}$.

It implies that ω_1^V is preserved in $V[G]$:

- Take a $\mathbb{P}$-name $\dot{f}$ for a function $\dot{f}\colon \omega \to \omega_1^V$.
- Pick a maximal antichain of conditions deciding $\dot{f}(n)$.
- Let α_n be the (countable) supremum of these values and $\alpha := \sup_{n<\omega} \alpha_n$.
- Then $\operatorname{ran}(\dot{f})$ is bounded below ω_1^V.

Forcing without choice

Forcing without choice

Steel and Van Wesep (1982) introduced forcing over models of determinacy.

Based on their work, Woodin developed $\mathbb{P}_{\max}$-forcing over models of determinacy.

$\mathbb{P}_{\max}$-forcing and its variants form a powerful machinery to construct models of choice by forcing over choiceless models.

Forcing without choice

Blass proved the following in extensions by a Levy collapse of an inaccessible: An ultrafilter on ω is Ramsey if and only if it is generic for $P(\omega)/\mathrm{fin}$ over $L(\mathbb{R})$.

This was extended in work of Laflamme and Todorcevic.

Forcing without choice

Some research on forcing over arbitrary choiceless models has been done.

Monro (1983) studied preservation of fragments of the axiom of choice.

Karagila, Schlicht (2020) studied under which circumstances $\mathrm{Add}(A, 1) = \{p \mid p\colon A \rightharpoonup 2 \text{ finite}\}$ adds new reals.

Chan, Jackson (2021) and Ikegami, Trang (2023) studied the preservation of the axiom of determinacy by forcing.

Forcing without choice

What can go wrong?

- Countably closed forcings can collapse ω_1 (folklore).
- Karagila, Schweber (2022): c.c.c. forcings can collapse ω_1.
- Karagila, Schilhan (2022): A forcing may add no new ω-sequences of ordinals, while it is not countably distributive.

  A forcing is called countably distributive if the intersection of countably many open dense sets is dense.
- Boolos (1974) or folklore: DC holds if and only if every structure has a countable elementary substructure. Thus the definition of proper forcing is not useful if DC fails.

Forcing without choice

The aim is to develop a general theory of forcing over choiceless models. We want to allow failures of even weak choice principles such as DC and AC_ω.

- What do classical forcings do over arbitrary choiceless models?
- Can one force anything interesting over arbitrary choiceless models?

Choiceless models

Example I: Cohen's first choiceless model

$\mathrm{Add}(\omega, \omega) := \{p \mid p \colon \omega \times \omega \to 2 \text{ finite}\}$.

Example

$\mathrm{Add}(\omega, \omega)$ adds a sequence $\vec{a} = \langle a_n \mid n \in \omega \rangle$ of Cohen reals.
Let $A := \{a_n \mid n \in \omega\}$ and $V(A)$ the least model $M \supseteq V$ of ZF with $A \in M$.

- DC fails in $V(A)$, since A does not have a countably infinite subset.

Example II: Gitik's model

Example

Gitik constructed a model of ZF where:

- All uncountable cardinals have countable cofinality.

The construction uses a proper class of strongly compact cardinals.

Remark

If ω_1 is singular, then AC_ω and therefore DC fails:

Proof.

Suppose not. Let $\vec{\alpha} = \langle \alpha_n \mid n \in \omega \rangle$ be cofinal in ω_1.

- Pick $\vec{f}$ with $f_n \colon \alpha_n \to \omega$ injective by AC_ω.

This yields an injective function $f \colon \omega_1 \to \omega$. □

Properties of forcings

Countably closed forcings

Definition

1. $\mathrm{Col}(\kappa, \lambda) := \{p \colon \alpha \to \lambda \mid \alpha < \kappa\}$.
2. $\mathrm{Col}_*(\kappa, \lambda) := \{(f, g) \mid f \in \mathrm{Col}(\kappa, \lambda),\ g \colon \mathrm{dom}(f) \to |\mathrm{dom}(f)| \text{ bijective}\}$.

$\mathrm{Col}(\kappa, \lambda)$ is ordered by reverse inclusion, while $\mathrm{Col}_*(\kappa, \lambda)$ is ordered by reverse inclusion in the first coordinate.

Remark

If ω_1 is singular, then $\mathrm{Col}(\omega_1, 2)$ is not countably closed.
But $\mathrm{Col}_*(\omega_1, 2)$ is countably closed.

Countably closed forcings

Theorem

TFAE for any set A of size at least 2, $\lambda \in \mathsf{Card}$ and $\mathbb{P} = \mathrm{Col}(\lambda^+, A)$:

1. $\mathsf{DC}_\lambda(A^\lambda)$.
2. $\mathbb{P}$ *is λ-distributive.*
3. $\mathbb{P}$ *does not change V^λ.*
4. $\mathbb{P}$ *preserves size and cofinality of all ordinals $\alpha \leq \lambda^+$.*
5. $\mathbb{P}$ *preserves λ^+ as a cardinal.*
6. $\mathbb{P}$ *forces that λ^+ is regular.*

The same equivalences hold for $\mathrm{Col}_*(\lambda^+, A)$.

Linked forcings

Let $\mathbb{C}^\kappa$ denote the finite support product of κ many Cohen forcings $\mathbb{C} = \{p \mid p\colon n \rightharpoonup 2, n \in \omega\}$.

Karagila observed that wellordered c.c.c. forcings such as $\mathbb{C}^\kappa$ preserve cardinals.

We can reduce finite support products and (uniform) iterations of σ-linked forcings fo $\mathbb{C}^\kappa$ to show they also preserve cardinals.

- A forcing $\mathbb{P}$ is called $\mathbb{Q}$-linked if there is a $\perp$-homomorphism from $\mathbb{P}$ to $\mathbb{Q}$.
- We equip each ordinal θ with the discrete partial order.

Linked forcings

We call a product or iteration of σ-linked forcings uniform if it comes with a sequence of names for linking functions.

Theorem

A uniform finite support product or iteration of σ-linked [$\mathbb{C}^\kappa$-linked?] forcings of length κ is $\mathbb{C}^\kappa$-linked.

Any $\mathbb{C}^\kappa$-linked forcing preserves cardinals.

This a special case of the following notion.

Narrow forcings

Definition

Suppose that $\mathbb{P}$ is a forcing and θ, ν are ordinals, where θ is infinite.

1. $\mathbb{P}$ is called (θ,ν)-narrow if for any ordinal $\mu \leq \nu$ and any sequence $\vec{f} = \langle f_i \mid i < \mu \rangle$ of partial $\|$-homomorphisms $f_i \colon \mathbb{P} \to \mathrm{Ord}$,
$$|\bigcup_{i<\mu} \mathrm{ran}(f_i)| \leq |\max(\theta,\mu)|.$$
2. $\mathbb{P}$ is called θ-narrow if it is (θ,ν)-narrow for all $\nu \in \mathrm{Ord}$. It is called narrow if it is ω-narrow.

We further call $\mathbb{P}$ uniformly (θ,ν)-narrow if there exists a function G_ν that sends each sequence $\vec{f} = \langle f_i \mid i < \mu \rangle$ of partial $\|$-homomorphisms $f_i \colon \mathbb{P} \to \mathrm{Ord}$,[1] where $\mu \leq \nu$, to an injective function
$$G_\nu(\vec{f}) \colon \bigcup_{i<\mu} \mathrm{ran}(f_i) \to \max(|\theta|,\mu).$$
It is called uniformly narrow if it is uniformly ω-narrow.

[1] We can assume $\mathrm{ran}(f_i)$ is an ordinal.

Narrow forcings

Lemma

Every (θ, ν)-narrow forcing $\mathbb{P}$ preserves all cardinals and cofinalities in the interval $(\theta, \nu^+]$.

Lemma

Suppose that θ, ν are cardinals, where θ is infinite, and $f\colon \mathbb{P} \to \mathbb{Q}$ is a $\perp$-homomorphism.

1. *$\mathbb{Q}$ is (θ, ν)-narrow, then $\mathbb{P}$ is (θ, ν)-narrow.*
2. *$\mathbb{Q}$ is uniformly (θ, ν)-narrow, then $\mathbb{P}$ is uniformly (θ, ν)-narrow.*

Narrow forcings

Theorem

Suppose that $\theta \leq \nu$ are infinite ordinals. Any uniform iteration of (θ, ν)-narrow forcings with finite support is again uniformly (θ, ν)-narrow.

This allows us to iterate a mix of Cohen forcing, Hechler forcing and random algebras while preserving all cardinals and cofinalities.

Specific forcings

Iterations of Hechler forcing

Theorem

Suppose that κ is a cardinal of uncountable cofinality. Then $\mathbb{H}^{(\kappa)}$ forces $\mathfrak{b} = \mathfrak{d} = \mathrm{cof}(\kappa)$.

Theorem

Suppose $\nu \geq \omega_1$ is multiplicatively closed and has countable cofinality. Any uniform iteration $\mathbb{P}_\nu$ of nontrivial forcings with finite support of length ν forces:

1. $\mathfrak{b} = \omega_1$ *if $\mathbb{P}_\nu$ preserves ω_1.*
2. $\mathfrak{d} \geq |\nu|$ *if $\mathbb{P}_\nu$ preserves $|\nu|$ and $\mathfrak{d}$ exists in the extension.*

In particular, this holds for $\mathbb{H}^{(\nu)}$.

Random algebras

An α-Borel code for a subset of 2^α is a subset of α that codes a set formed from basic open subsets of 2^α via complements and countable unions. Let $2^{(\alpha)} = \{f \mid f\colon \alpha \rightharpoonup 2 \text{ finite.}\}$.

$\mathbb{R}_\alpha$ denotes the forcing that consists of all Borel codes for subsets of 2^α ordered by $\leq$. The quotient of $\mathbb{R}_\alpha$ by $=_\mu$ with the operations induced by $\vee$, $\wedge$ and $-$ is a Boolean algebra.

A forcing is called complete if every subset has a supremum. To show $\mathbb{R}_\alpha$ is complete, we associate to every $A \in \mathbb{R}_\alpha$ its footprint $\mathrm{f}_A = \langle \mathrm{f}_{A,t} \mid t \in 2^{(\alpha)} \rangle$, where $\mathrm{f}_{A,t}$ denotes the relative measure:

$$\mathrm{f}_{A,t} := \frac{\mu([p] \cap N_t)}{\mu(N_t)}.$$

Let $\mathrm{f}_A \leq \mathrm{f}_B$ if $\mathrm{f}_{A,t} \leq \mathrm{f}_{B,t}$ for all $t \in 2^{(\alpha)}$. Note that $A \leq B$ if and only if $\mathrm{f}_A \leq \mathrm{f}_B$.

Definition

Suppose that $\vec{\mathfrak{f}} = \langle \mathfrak{f}_s \mid s \in 2^{(\alpha)} \rangle$ is a sequence in $\mathbb{R}$ and $x \in 2^\alpha$.

1. For any $\epsilon > 0$, x is called an ϵ-density point of $\mathfrak{f}$ if

$$\exists s \, \forall t \supseteq s \; \mathfrak{f}_t > 1 - \epsilon.$$

2. x is called a density point of $\mathfrak{f}$ if it is an ϵ-density point of $\mathfrak{f}$ for all $\epsilon \in \mathbb{Q}^+$.

The α-Borel code induced by 2 is denoted $D(\mathfrak{f})$.

Random algebras

To construct a least upper bound, we first form the least upper bound of the footprints: let $\mathfrak{f}_{X,t} := \sup_{A \in X} \mathfrak{f}_{A,t}$ for each $t \in 2^{(\alpha)}$ and

$$\mathfrak{f}_X := \langle \mathfrak{f}_{X,t} \mid t \in 2^{(\alpha)} \rangle.$$

Lemma

1. *In any outer model W of V such that α is countable in W, $D(\mathfrak{f}_X)$ is a least upper bound for X.*
2. *$\mathbb{R}_\alpha$ is complete. More precisely, for any subset X of $\mathbb{R}_\alpha$ the reduct of $D(\mathfrak{f}_X)$ is a least upper bound for X.*

The reduct is defined by induction on the rank by reducing each union by a countable one.

Using completeness, we can show random algebras are uniformly narrow.

Future directions

Changing the theory

Theorem (Woodin 2013)

The Cohen and random models over any choiceless model have different first order theories.

This is the only such example of two models up to now. We have not separated the Hechler from the Cohen and random models.

Theorem

Adding many Cohen reals over Gitik's model changes its first order theory.

Grounds

The ground axiom (introduced by Reitz 2014) states that the universe has no nontrivial ground for set forcing. We aim to analyse the grounds of choiceless models.

The following is motivated by questions of Usuba and Larson.

Question

Suppose there is a proper class of measurable cardinals. Does the Chang model $L(\mathrm{Ord}^\omega)$ have a nontrivial ground?

We aim to first understand Solovay's model.

Question

What are precisely the grounds of Solovay's model?

Thank you!