Implicitly definable sets and large cardinals

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Overview

Topic:

- Hamkins and Leahy (2014) introduced implicitly definable sets of ordinals over L.
- Carl, Welch and Schlicht (2018) introduced recognisable sets of ordinals.
- Aim: study the interplay with large cardinals.

Based on joint work with Carl and Welch:

- Merlin Carl, Philipp Schlicht and Philip Welch: *Recognisable sets and measurable cardinals*, 13 pages, in preparation

Implicitly definable functions

An implicit definition of a function f is a decision procedure to answer the question:

- Is (x, y) is in the graph of f?

Calculating $f(x)$ may be harder than checking whether $f(x) = y$.

Example

$x^2 + y^2 = 1$ versus $f(x) = (1 - x^2)^{\frac{1}{2}}$

Implicitly definable sets

Our notion of algorithm is relative constructibility for sets of ordinals with ordinal parameters.

The pointwise version of the above problem asks:

- Is it harder to construct a set y of ordinals in $L(x)$ than checking a condition in $L(x, y)$?
- Is it harder to construct a set y of ordinals in L than checking a condition in $L(y)$?

The implicitly definable universe

Definition (Hamkins, Leahy 2014)

Suppose that M is a class.

- A subset X of M is called implicitly definable over M if for some first-order formula $\varphi(.)$ with parameters in M, X is unique with

$$(M, \in, X) \models \varphi(X).$$

- Let $P_{\mathrm{imp}}(M)$ denote the class of subsets of M which are implicitly definable over M.
- Let $\mathrm{Imp}_0 = \emptyset$, $\mathrm{Imp}_{\alpha+1} = P_{\mathrm{imp}}(\mathrm{Imp}_\alpha)$ and $\mathrm{Imp}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{Imp}_\alpha$ for limits λ.
- $\mathrm{Imp} = \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{Imp}_\alpha$ is called the implicitly definable universe.

The implicitly definable universe

Proposition (Hamkins, Leahy 2014)
Imp is a model of ZF with $L \subseteq \mathrm{Imp} \subseteq \mathrm{HOD}$.

They asked:

- Which large cardinals are absolute to Imp?
- Can Imp have measurable cardinals?
- Can we put arbitrary sets into the Imp of a suitable forcing extension?

Recognisable sets

Definition (Carl, Schlicht, Welch 2018)

A set x of ordinals is called recognisable if there is a first-order formula $\varphi(y, \alpha)$ with an ordinal parameter α such that x is the unique set y of ordinals with

$$L[y] \models \varphi(y, \alpha).$$

Recognisable sets need not be in L.

Example

$0^{\#}$, and any Π^1_2-singleton x, is recognisable with parameter ω_1, since Π^1_2-truth is absolute between $L[x]$ and V.

These examples are analogous to the lost melody phenomenon in infinite time computation: a real may be decidable, but not writable.

The recognisable universe

The class of recognisable sets is not necessarily constructibly closed.

Example

A Cohen real x over L cannot be recognisable. Otherwise, all Cohen reals over L extending a specific finite sequence would satisfy the formula recognising x.

But $0^{\#}$ is recognisable and constructs Cohen reals over L.

The recognisable universe R denotes the constructible closure:

$$\mathrm{R} = \bigcup_{x \text{ is recogn.}} L[x].$$

R equals the class of sets coded by recognisable sets, via the Mostowski collapse.

Recognisable $\leftrightsquigarrow$ implicitly definable

Proposition (Carl, Schlicht, Welch 2018)

The constructible closures of the following classes are equal:

(1) Recognisable sets

(2) Implicitly definable sets over L

To show (1) $\Rightarrow$ (2), one finds a set A of ordinals coding the $L_\alpha[x]$-hierarchy that is implicitly definable over L.

Every set implicitly definable over L is also recognisable, but $0^\#$ is recognisable and it's open whether it is implicitly definable over L.

Corollary

R *is a subclass of* Imp.

The implicitly definable universe is variable

Theorem (Groszek, Hamkins 2017)

Each of the following statements is consistent:

- $\mathrm{Imp} \models \neg\mathsf{CH}$
- $\mathrm{Imp} \neq \mathsf{HOD}$
- $\mathrm{Imp}^{\mathrm{Imp}} \neq \mathrm{Imp}$

Background: inner models built from strong logics

Replace first-order logic by a stronger logic to obtain variants of L:

- HOD arises from second-order logic (Myhill, Scott 1971)
- Chang's model $L(\mathrm{Ord}^\omega)$ arises from $\mathcal{L}_{\omega_1,\omega_1}$ (Chang 1971)
- $L[\mathrm{Card}]$
- $L[\mathrm{Cof}\,\omega]$

While HOD is too variable to achieve a complete analysis, the remaining models have been analysed, assuming large cardinals. (Woodin 2004, Welch 2019, Magidor, Kennedy, Väänänen 2020)

In particular, their first-order theories are absolute, assuming a proper class of Woodin cardinals.

Recall that BPFA denotes the forcing axiom for ω_1-many predense sets, each of size at most ω_1.

Caicedo and Velickovic isolated a consequence T of BPFA in H_{ω_2} that fixes the power set of ω_1.

Theorem (Caicedo, Velickovic 2006)

Suppose that M is an inner model with $\omega_2^M = \omega_2$ and T holds in both $H_{\omega_2}^M$ and H_{ω_2}. Then $H_{\omega_2}^M = H_{\omega_2}$.

Forcing axioms

Larson (2008) obtained a wellorder of H_{ω_2} that is definable without parameters over H_{ω_2} from strong forcing axioms.

Let A be the subset of ω_2 that lists $P(\omega_1)$ with respect to this wellorder. Then A is recognisable.

It follows that assuming large cardinals, every set can be put into the Imp of a generic extension.

$\mathbb{R}$ in the presence of measurables

$L[U]$ denotes a model with a normal ultrafilter U on κ over $L[U]$.

Kunen proved that $L[U]$ is unique: if $L[U']$ is such that U' is a normal ultrafilter on κ over $L[U']$, then $U = U'$.

Proof sketch:

- Take iterated ultrapowers of $L[U]$ and $L[U']$ up to a regular λ.
- Both images of U and U' will equal the club filter on λ.
- Forming Skolem hulls shows $U = U'$.

This implies: $L_\lambda[U]$, coded by its canonical wellorder, is recognisable from κ and λ.

Similar arguments work for $L[U_0, \ldots, U_n]$. Hence $\mathbb{R} = V$ in these models.

$\mathbb{R}$ in the presence of measurables

By a fine-structural model, we mean a transitive model $M = J_\alpha[E]$ with a coherent extender sequence E in the sense of Zeman 2002, Chapter 4, that does not contain measures of order 1.

Lemma (Carl, Schlicht, Welch 2023)

Suppose that M and N are iterable small fine-structural models with the same measurable cardinals $\kappa_0 < \cdots < \kappa_n$ of the same height, $\mathrm{cof}^V(\kappa_i) > \omega$ and $\kappa_i^{+M} = \kappa_i^{+N} = \kappa_i^+$ for all $i \leqslant n$.

- *If $\mathrm{l}(E^M) = \mathrm{l}(E^N) = \lambda^+$ for some λ such that λ^+ is a successor cardinal in both M and N, then $M = N$.*

R in the presence of measurables

If M is a fine-structural model of height $\lambda \in$ Card, let X_M denote the unique subset of λ that codes M via its canonical wellorder.

Proposition (Carl, Schlicht, Welch 2023)

Suppose that V is an iterable small fine-structural model with the measurable cardinals $\kappa_0 < \cdots < \kappa_n$ and $\mathrm{l}(E^V) = \lambda$. Then $X_{V||\lambda}$ is recognisable from κ_i, κ_i^+ for $i \leqslant n$ and

1. λ, if λ is a successor cardinal.
2. λ and some ordinal α with $\lambda \leqslant \alpha < \lambda^+$.

In particular, $\mathrm{R} = V$.

R in the presence of measurables

M^{meas} denotes the least fine-structural model with infinitely many measurable cardinals with supremum κ. Write $L[\![X]\!] := \bigcup_{x \in X} L(x)$.

The above results show one inclusion of the next theorem.

Theorem (Carl, Schlicht, Welch 2023)
In M^{meas}, $\mathrm{R} = L[\![H_\kappa]\!]$.

The reverse inclusion is shown via the analysis of HOD in iterated ultrapowers of Dehornoy 1978.

What's next?

Problem

Can we compute R inner models with larger cardinals?

- We aim to do this below a Woodin cardinal.

M_1 is a canonical inner model with a Woodin cardinal. We obtain M^∞ by taking all iterated ultrapowers of M_1 with the measure on its least measurable cardinal.

Theorem (Carl, Schlicht, Welch 2018)

All recognisable subsets of countable ordinals are elements of M^∞.

- This fails for subsets of ω_2 by results of Caicedo and Velickovic.
- However, one can show that a generic version $\mathrm{R}^{\mathrm{gen}}$ of R equals M^∞.

Problem

Do the levels of Imp roughly correspond to Woodin cardinals?