Iterated forcing and determinacy

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I. Introduction

Two themes

Problem

Which iterated forcings preserve projective determinacy?

This talk is based on joint work with Jonathan Schilhan (Leeds) and Johannes Schürz (Vienna).

- Jonathan Schilhan, Philipp Schlicht, Johannes Schürz:
  Iterated forcing, determinacy and regularity, 32 pages, in preparation

Two themes

Iterated forcing is the main tool used to prove independence results for arbitrary sets of reals.

Example

The Borel conjecture states that every strong measure 0 set is countable.

Recall that for any sequence of positive reals ϵ_n, a strong measure 0 set is covered by a sequence of balls of radii ϵ_n.

- CH implies that the Borel conjecture fails
- Laver (1976) showed that the Borel conjecture holds after iterating Laver forcing

Two themes

Projective determinacy is an important axiom used to study definable sets of reals beyond Borel sets. Some consequences:

- Measurability
  - Lebesgue measurability of all projective sets
- Nonexistence
  - No projective wellordering of the reals
  - No projective selector for equality up to finite error (E_0)
- Structure
  - Projective uniformisation of all projective relations

II. Forcings on the reals and analytic determinacy

Test question

Question

Does Cohen forcing preserve analytic determinacy?

We do not know of a direct proof using the definition of analytic determinacy.

What was known

It is easy to destroy determinacy if ω_1 can be collapsed. We will thus assume $\mathbb{P}$ is proper.

In fact, we will always assume stronger forms of properness. It is open whether properness suffices.

It is further natural to assume $\mathbb{P}$ is a projective forcing on the reals.

The complexity of the forcing should be approximately the same as the level of projective determinacy:

Theorem (David 1978)

It is consistent that there exists a Σ^1_3-definable c.c.c. forcing that destroys analytic determinacy.

What was known

Σ^1_3-absoluteness is closely related to this, since it often follows from proofs of the preservation of analytic determinacy.

Theorem (Woodin 1982)

Analytic determinacy (actually uniformisation up to meager resp. null) implies Σ^1_3-absoluteness for Cohen and random forcing.

Problem

Does every Borel proper forcing preserve analytic determinacy?

This is a slight variant of a question of Ikegami (2010) who asked this for absolutely Δ^1_2 proper tree forcings.

What was known

Theorem (S. 2014)

Any Σ^1_2 absolutely c.c.c. forcing preserves Π^1_n-determinacy for each $n \geq 1$.

Theorem (Castiblanco, S. 2020)

Sacks, Mathias, Laver and Silver forcing preserve Π^1_n-determinacy for each $n \geq 1$.

Background

Theorem (Levy, Solovay 1967)

If κ is measurable and $\mathbb{P}$ is a forcing of size $|\mathbb{P}| < \kappa$, then κ remains measurable in any $\mathbb{P}$-generic extension $V[G]$ of V.

Proof sketch.

Lift $j\colon V \to N$ to $j^*: V[G] \to N[G]$ by letting $j^*(\sigma^G) = j(\sigma)^G$. □

Theorem (Foreman 2013)

Generic supercompactness of ω_1 is preserved by all proper forcings.

Analytic determinacy and sharps

Theorem (Martin 1970, Harrington 1978)
The following conditions are equivalent:

1. $x^{\#}$ *exists for all reals* x
2. *Analytic determinacy*

Analytic determinacy is preserved by Cohen forcing by lifting $j\colon L[\sigma] \to L[\sigma]$ to $L[\sigma][G]$.

A strong variant of proper

Definition (Goldstern 1992)
A Suslin forcing is a forcing $(\mathbb{P}, \leq)$ on the reals such that $\leq$ is analytic.

Definition (following Judah, Shelah 1988)
Let $\mathbb{P}$ be a forcing on the reals.

1. A countable transitive model N of a large fragment of ZFC is called a candidate for $\mathbb{P}$ if $\mathbb{P} \cap N \in N$.
2. $\mathbb{P}$ is called proper-for-candidates if for every candidate N for $\mathbb{P}$ and every $p \in \mathbb{P} \cap N$ there exists an $(N, \mathbb{P})$-generic condition $q \leq p$.

We similarly define $\mathbb{M}$-proper if $\mathbb{M}$ is a family of transitive models of ZFC^-.

Capturing

We isolated a weak form a properness that suffices for the preservation of sharps:

Definition (Castiblanco, S. 2020)

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcings and $\mathbb{Q}$ is amenable to each $L(x)$. We say that $\mathbb{P}$ is L-captured by $\mathbb{Q}$ if the following holds for any $\mathbb{P}$-name τ for a real

" If H is $\mathbb{P}$-generic over V, then τ^H is contained in a $\mathbb{Q} \cap L(y)$-generic extension of some $L(y)$. "

Equivalently: For any $p \in \mathbb{P}$ and any real x, there exists a real y with $x \in L(y)$ such that some $q \leq_{\mathbb{P}} p$ forces:

"There exists a $\mathbb{Q} \cap L(y)$-generic filter g over $L(y)$ with $\tau \in L(y)[g]$."

$\mathbb{M}$-captured is defined similarly for any operator $\mathbb{M}(x)$ instead of $L(x)$, i.e. a function $\mathbb{M}$ that sends each real x to a structure $\mathbb{M}(x) = (M(x), \in, E)$ such that

- $x \in M(x)$, $M(x)$ is transitive, $M(x) \models \mathrm{ZFC}^-$ and $M(x)$ is E-amenable.

Capturing

Any proper-for-candidates Suslin forcing is L-captured (by some forcing). This includes most classical proper forcings which add a real.

Any L-captured forcing on the reals implies preservation of analytic determinacy.

Capturing

Lemma (Schilhan, Schürz, S.)

Suppose that ω_1 is inaccessible to the reals, $\bar{\mathbb{P}} = (P, \vec{\lesssim})$ is an Axiom A forcing on the reals and the following hold for any real x:

1. $\mathbb{P} \cap L[x] \in L[x]$ *satisfies Axiom A in* $L[x]$.
2. *For any subset $A \in L[x]$ of $\mathbb{P}$ that is countable in $L[x]$ and for any $p \in \mathbb{P}$, the statements*

   "A is an antichain"

   "A is predense below p"

   are absolute between $L[x]$ and V.

Then $\mathbb{P}$ is L-captured by $\mathbb{P}$.

A similar result holds for arbitrary operators.

III. Forcings on the reals and projective determinacy

Iterable structures

Work with transitive structures $(M, \in, E)$, where

- $(M, \in) \models \mathrm{ZFC}^-$
- E is an M-amenable sequence of extenders, i.e., directed systems of ultrafilters.
- All extenders in E, except possibly the last one, are elements of M

An iteration (tree) on $(M, \in, E)$ is formed along a tree order T such that an extender can be applied to a different model than the last one.

Definition

$(M, \in, E)$ is called ω_1-iterable if there exists a strategy choosing branches such that all ultrapowers in countable iteration trees on M using E and its images are wellfounded.

Definition

We call $(M, \in, E)$ n-tall if M has n Woodin cardinals and a measure above them, witnessed by E.

An operator $\mathbb{M}$ is called n-tall if each $\mathbb{M}(x)$ is n-tall.

Iterable structures

Definition

Suppose that M is a model, $\mathbb{N}$ an operator and t a term.

1. An iteration S on M of limit length is called $(\mathbb{N}, t)$-*good* if the following statements hold for $\sigma_t(S, y) := t_S^{\mathbb{N}(y)}$, where y is any real with $S \in \mathbb{N}(y)$:
   (a) $\sigma_t(S, y)$ exists.
   (b) $\sigma_t(S, y)$ is independent of the choice of y.
   (c) $\sigma_t(S, y)$ is a wellfounded branch in S.

   Let $\sigma_t(S) := \sigma_t(S, y)$ in this case.
2. M is called $\mathbb{N}$-iterable if there is a term t such that σ_t is a (nice) iteration strategy for M. In particular, for all iterations T on M of limit length, the direct limit along $\sigma_t(T)$ is wellfounded if all proper initial segments S of T of limit length are $(\mathbb{N}, t)$-good and pick $\sigma_t(S)$ at that stage.
3. An operator $\mathbb{M}$ is called $\mathbb{N}$-*iterable* if there is a term t that witnesses each $\mathbb{M}(x)$ to be $\mathbb{N}$-iterable.

An operator $\mathbb{M}$ is called stably $\mathbb{N}$-iterable if the above holds for all small generic extensions $\mathbb{N}(y)[g]$ of $\mathbb{N}(y)$.

Projective determinacy

The following can be extracted from work of Neeman, Woodin and others:

Corollary

The following statements are equivalent for any n:

1. Π^1_{n+1}-*determinacy.*
2. *There exists an n-tall stably $\mathbb{M}$-iterable operator $\mathbb{M}$.*

For example, the $\mathbb{M}_n^\#$-operator from inner model theory is stably $\mathbb{M}_k^\#$-iterable for any $k \geq n-1$.

Preservation of projective determinacy

The notion of stable iterability is defined so that one can extend the operator to generic extensions.

Proposition (Schilhan, Schürz, S.)

Suppose that $\mathbb{M}$ is any stably $\mathbb{M}$-iterable $\mathbb{P}$-amenable operator and $\mathbb{P}$ is $\mathbb{M}$-captured.

1. In any $\mathbb{P}$-generic extension, $\mathbb{M}$ can be extended to an $\mathbb{M}^*$-iterable operator $\mathbb{M}^*$.
2. If $\mathbb{M}$ is n-tall, then $\mathbb{M}^*$ is n-tall.

It follows for example that any proper-for-candidates Suslin forcing preserves projective determinacy level-by-level.

Corollary (Schilhan, Schürz, S.)

Any Σ^1_3 proper-for-candidates forcing preserves Σ^1_2-determinacy.

- This contrasts David's result that Σ^1_3 c.c.c. forcings can destroy analytic determinacy.

IV. Iterating forcings on the reals

Background

Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha < \kappa \rangle$ is a countable support iteration of proper-for-candidates Suslin forcings (in fact, they assumed $\perp$ is analytic as well) 0 of countable length α. Suppose M is a countable transitive model.

- Judah, Shelah and Goldstern defined a procedure to translate a $\mathbb{P}$-generic filter G over V to a filter G^M on $\mathbb{P}^M$.

Theorem (Judah, Shelah 1988, Goldstern, Shelah 1992)

Over Solovay models, countable support iterations of Suslin proper-for-candidates forcings preserve that all projective sets have the Baire property.

- Main application: force the Borel conjecture over Solovay's model while preserving the Baire property for projective sets.

Iterated forcing

Theorem (Schilhan, Schürz, S.)

Suppose that ω_1 is inaccessible in $L(x)$ for every real x. Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha < \kappa \rangle$ is a countable support iteration of proper-for-candidates Suslin forcings such that for every $\alpha < \kappa$,

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{P}}_\alpha \text{ is proper-for-candidates in any small generic extension of any } L(x).$$

Then $\mathbb{P}$ is L-captured by a countable support iteration $\mathbb{Q}$ of proper-for-candidates Suslin forcings of countable length.

In particular, forcing with $\mathbb{P}$ preserves analytic determinacy.

The forcing $\mathbb{Q}$ is constructed in a concrete way.

Example

If $\mathbb{P}$ is an iteration of Sacks forcing, then $\mathbb{Q}$ is an iteration of Sacks forcing as well.

Iterated forcing

Theorem (Schilhan, Schürz, S.)

Suppose that n is even and $\mathbb{M}$ is an n-tall stably iterable operator. Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha < \kappa \rangle$ is a countable support iteration of $\mathbb{M}$-proper $\boldsymbol{\Sigma}^1_{n+2}$-forcings such that for every $\alpha < \kappa$,

> $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{P}}_\alpha$ *is proper for $\boldsymbol{\Sigma}^1_{n+2}$-correct models in any small generic extension of any $\mathbb{M}(x)$.*

Then $\mathbb{P}$ is L-captured by a countable support iteration of countable length of forcings $\mathbb{Q}$ that are proper for $\boldsymbol{\Sigma}^1_{n+2}$-correct models.

In particular, forcing with $\mathbb{P}$ preserves $\boldsymbol{\Pi}^1_{n+1}$-determinacy.

Corollary (Schilhan, Schürz, S.)

The combination of the Borel conjecture and $\boldsymbol{\Pi}^1_n$-determinacy is consistent, if the latter is consistent.

V. More results

Generic absoluteness, adding equivalence classes

Theorem (Schilhan, Schürz, S.)

Suppose that ω_1 is inaccessible to the reals and $\mathbb{P}$ is L-captured if $n = 0$ and $\mathbb{P}$ is $\mathbb{M}$-captured if $n > 0$, where $\mathbb{M}$ is an n-tall stably iterable operator. Then

$$V \prec_{\Sigma^1_{n+3}} V^{\mathbb{P}}.$$

Building on arguments of Hjorth and Magidor about not adding new classes to thin equivalence relations, we get for example:

Proposition (Schilhan, Schürz, S.)

Suppose analytic determinacy holds and $\mathbb{P}$ is a proper forcing on the reals such that $\mathbb{P}$ and $\mathbb{P} \times \mathbb{P}$ are captured. Then $\mathbb{P}$ does not increase u_2.

Theorem (Schilhan, Schürz, S.)

Suppose analytic determinacy holds and G is an absolutely $\mathbf{\Delta}^1_3$ thin graph on the reals. After forcing with a countable support iteration of Sacks forcing, any new real has an edge to some ground model real.

It follows from results of Pawlikowski (1986), Judah and Shelah (1989) that

1. Cohen forcing preserves "all $\mathbf{\Delta}^1_2$ sets have the Baire property."
2. Random forcing preserves "all $\mathbf{\Delta}^1_2$ sets are Lebesgue measurable."

Theorem (Schilhan, Schürz, S.)

Any forcing that satisfies a uniform version of capturing for Cohen forcing preserves that all $\mathbf{\Delta}^1_2$ sets have the Baire property.

- For example, countable support products and iterations of Sacks forcing satisfy this.

VI. Open questions

Open questions

Question

Do the above forcings preserve determinacy in $L(\mathbb{R})$?

Question

Does proper imply proper-for-candidates and Axiom A for Borel forcings, assuming projective determinacy?

This is related to Ikegami's question whether absolutely Δ^1_2 proper tree forcings preserve analytic determinacy by

Question

Can Mathias forcing with an ultrafilter destroy analytic determinacy?

Thank you!