Interplay of determinacy and forcing

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Two themes

Projective determinacy is an important axiom used to study definable sets of reals beyond Borel sets. Consequences:

- Regularity
  - Lebesgue measurability of all projective sets
- Nonexistence
  - No projective wellordering of the reals
  - No selector for equality up to finite error (E_0)
- Structure
  - Projective uniformisation of all projective relations

Two themes

Iterated forcing is the main tool used to study the independence of properties of sets of reals.

Example

The Borel conjecture states that every strong measure 0 set is countable.

Recall that for any sequence of positive reals ϵ_n, a strong measure 0 set is covered by a sequence of balls of radii ϵ_n.

- CH implies that the Borel conjecture fails
- Laver (1976) showed that the Borel conjecture holds after iterating Laver forcing

Two themes

We connect these two approaches by studying the

Problem

Which iterated forcings preserve projective determinacy?

Joint work with Jonathan Schilhan (Leeds) and Johannes Schürz (Vienna).

- Jonathan Schilhan, Philipp Schlicht, Johannes Schürz:
  The interplay of iterated forcing with determinacy and regularity
  33 pages, in preparation

Test question

Question
Does Cohen forcing preserve analytic determinacy?

We do not know of a direct proof using the definition of analytic determinacy.

Determinacy

Fix a subset A of 2^ω. In the game $G(A)$, two players I and II alternate playing moves with values 0 and 1.

I	x_0		x_2		x_4		x_6		...
II		x_1		x_3		x_5		x_7	...

II wins the run $\iff x = \langle x_n \mid n \in \omega \rangle \in A$

$G(A)$ is called determined if I or II has a winning strategy.

Theorem (Martin 1975)
All Borel sets are determined.

Projective determinacy (PD) is the statement that all projective sets are determined.

What was known

It is easy to destroy determinacy if ω_1 can be collapsed. We will thus assume $\mathbb{P}$ is proper.

In fact, we will always assume stronger forms of properness. It is open whether properness suffices.

It is further natural to assume $\mathbb{P}$ is a projective forcing on the reals. The complexity of the forcing is approximately the same as the level of projective determinacy.

Theorem (David 1978)

It is consistent that there exists a Σ^1_3-definable c.c.c. forcing that destroys analytic determinacy.

What was known

Analytic determinacy is closely linked with Σ^1_3-absoluteness.

Theorem (Woodin 1982)

Analytic determinacy (actually uniformisation up to meager resp. null) implies Σ^1_3-absoluteness for Cohen and random forcing.

The proofs of preservation of projective determinacy also show projective absoluteness.

What was known

Problem

Does every Borel proper forcing preserve analytic determinacy?

This is really a question of Ikegami (2010). He asked this for absolutely Δ^1_2 proper tree forcings.

Theorem (S. 2014)

Any Σ^1_2 absolutely c.c.c. forcing preserves Π^1_n-determinacy for each $n \geq 1$.

Theorem (Castiblanco, S. 2020)

Sacks, Mathias, Laver and Silver forcing preserve Π^1_n-determinacy for each $n \geq 1$.

Background

A cardinal κ is measurable if the following equivalent conditions hold:

- There is a non-principal $<\kappa$-complete ultrafilter on κ.
- There is an elementary embedding $j : V \to N$ to some transitive model N with $\mathrm{crit}(j) = \kappa$.

Theorem (Levy, Solovay 1967)

If κ is measurable and $\mathbb{P}$ is a forcing of size $|\mathbb{P}| < \kappa$, then κ remains measurable in any $\mathbb{P}$-generic extension $V[G]$ of V.

Proof sketch.

Lift $j : V \to N$ to $j^* : V[G] \to N[G]$ by letting $j^*(\sigma^G) = j(\sigma)^G$. □

Many variants of this theorem are known, for example for strong, Woodin, supercompact cardinals.

Background

Some large cardinal properties of small cardinals are preserved by sufficiently nice forcings.

Theorem (Foreman 2013)

Generic supercompactness of ω_1 is preserved by all proper forcings.

Sharps

Definition (Silver, folklore)

$0^\#$ exists if (equivalently) each of the following objects exist:

1. An uncountable set of ordinals which are order-indiscernible over L.
2. A non-trivial elementary embedding $j : L \to L$.
3. A countable structure $(L_\alpha, \in, U)$ such that
    - $(L_\alpha, \in)$ is a model of ZFC$^-$ with a largest cardinal κ,
    - $(L_\alpha, \in, U) \models \Sigma_0$-separation + U is a $<\kappa$-complete ultrafilter on κ
    - All iterated ultrapowers of $(L_\alpha, \in, U)$ are wellfounded.

    The least such structure is denoted $M_0^\#$ or $0^\#$.

More generally, $x^\#$ is defined for any real x by replacing L with $L[x]$.

A measurable cardinal implies that $x^\#$ exists for all reals x.

Analytic determinacy and sharps

Theorem (Martin 1970, Harrington 1978)

The following conditions are equivalent:

1. $x^{\#}$ *exists for all reals* x
2. *Analytic determinacy*

Cohen forcing

Proposition (folklore)

Cohen forcing preserves analytic determinacy.

Proof.

Suppose:

- $x^{\#}$ exists for all reals x.
- $V[G]$ is a Cohen extension of V. Let x denote the Cohen real.
- σ is a name for a new real. We can assume σ is a nice name.

The name σ is essentially a real, since Cohen forcing has the c.c.c. Thus $\sigma^{\#}$ exists. Hence there is a nontrivial elementary embedding $j\colon L[\sigma] \to L[\sigma]$.

> *x is Cohen generic over $L[\sigma]$, since Cohen forcing has the c.c.c. and is Σ^1_2-definable.*

We can lift j to $j^*: L[\sigma][G] \to L[\sigma][G]$ as in the Levy-Solovay theorem. Since the new real $\sigma^G \in L[\sigma][G]$, this yields $(\sigma^G)^{\#}$ in $V[x]$. □

Sacks forcing

Proposition (Castiblanco, S. 2020)

Sacks forcing $\mathbb{P}$ preserves analytic determinacy.

Proof sketch.

Again, we obtain a small $\mathbb{P}$-name σ and a nontrivial elementary embedding $j \colon L[\sigma] \to L[\sigma]$.

> *Force over $L[\sigma]$ in V with finite subtrees of $2^{<\omega}$ ordered by end extension.*

This adds a perfect tree T such that all its branches are Cohen reals over $L[\sigma]$. This remains true in generic extensions of V.

Force with $\mathbb{P}$ below $T \in \mathbb{P}$ over V. Let x denote the Sacks real.

> *Then x is Cohen generic over $L[\sigma]$.*

Again, we can lift j to $j^* \colon L[\sigma][x] \to L[\sigma][x]$ and obtain $(\sigma^x)^\#$ in $V[x]$. □

Definition (Shelah 1980)

1. Suppose that $\mathbb{P}$ is a forcing, $\dot{G}$ is a $\mathbb{P}$-name for the $\mathbb{P}$-generic filter and M is a model. A condition p is called $(M,\mathbb{P})$-generic if

$$q \Vdash \text{“}\dot{G} \cap N \text{ is } (\mathbb{P} \cap N)\text{-generic over } N\text{”}.$$

2. $\mathbb{P}$ is proper if for sufficiently large regular θ, there exists a club of countable $M \prec H_\theta$ with the following property.

“For every $p \in \mathbb{P} \cap M$, there exists some $(M,\mathbb{P})$-generic $q \leq p$.”

Suslin forcing

Definition (Goldstern 1992)

1. A Suslin forcing is a forcing on the reals with a Σ^1_1 definition of $\leq$.
2. A strongly Suslin forcing is a Suslin forcing with a Σ^1_1 definition of $\perp$.

Definition (following Judah, Shelah 1988)

Let $\mathbb{P}$ be a forcing on the reals, e.g. a Suslin forcing.

1. A countable transitive model N of a large fragment of ZFC is called a candidate for $\mathbb{P}$ if $\mathbb{P} \cap N \in N$.
2. $\mathbb{P}$ is called proper-for-candidates[1] if for every candidate N for $\mathbb{P}$ and every $p \in \mathbb{P} \cap N$ there exists an $(N, \mathbb{P})$-generic condition $q \leq p$.

[1] Originally called proper Suslin forcing

Definition (Castiblanco, S. 2020)

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcings and $\mathbb{Q}$ is amenable to each $L(x)$. We say that $\mathbb{P}$ is captured by $\mathbb{Q}$ over L if the following holds for any $\mathbb{P}$-name τ for a real

" If H is $\mathbb{P}$-generic over V, then τ^H is contained in a $\mathbb{Q} \cap L(y)$-generic extension of some $L(y)$. "

Equivalently: For any $p \in \mathbb{P}$ and any real x, there exists a real y with $x \in L(y)$ such that some $q \leq_{\mathbb{P}} p$ forces:

"There exists a $\mathbb{Q} \cap L(y)$-generic filter g over $L(y)$ with $\tau \in L(y)[g]$."

The same makes sense for any operator $\mathbb{M}(x)$ instead of $L(x)$:

Definition

An operator is a function $\mathbb{M}$ that sends each real x to a structure $\mathbb{M}(x) = (M(x), \in, E)$ such that

- $x \in M(x)$, $M(x)$ is transitive, $M(x) \models \text{ZFC}^-$ and $M(x)$ is E-amenable.

Capturing

All proper-for-candidates Suslin forcings are captured over L. This includes most classical proper forcings which add a real.

Capturing over L implies preservation of analytic determinacy, for proper forcings on the reals.

Iterable structures

One can reformulate the above preservation proofs using iterable structures $(M, \in, U)$ instead of elementary embeddings $j \colon L(\sigma) \to L(\sigma)$.

The Cohen or Sacks real x over V is generic over some $(M_0, \in, U)$. The iteration lifts step by step to

$$M_0[x] \to M_1[x] \to \cdots \to M_\alpha[x] \to \dots$$

More generally, work with transitive structures $(M, \in, E)$ where

- $(M, \in) \models \mathrm{ZFC}^-$
- E is an M-amenable sequence of (partial) extenders
- All extenders in E, except possibly the last one, are elements of M

A extender is a directed system of ultrafilters.

Definition

We call $(M, \in, E)$ n-tall if M has n Woodin cardinals and a measure above them, witnessed by E.

An operator $\mathbb{M}$ is called n-tall if each $\mathbb{M}(x)$ is n-tall.

Iterable structures

Why is it not so easy to preserve more complex iterable structures?

An iteration tree on $(M, \in, E)$ is formed along a tree order T such that an extender can be applied to a different model than the last one.

Definition

$(M, \in, E)$ is called ω_1-iterable if there exists a strategy choosing branches such that all ultrapowers in countable iteration trees on M using E and its images are wellfounded.

Small model characterisations

Let $\Phi(M)$ be a property of models of the form $L_\alpha[X]$, where X is a set and $\alpha \leq \mathrm{Ord}$.

Definition

S_Φ states that every real is an element of a transitive class model M of ZFC$^-$ as above with $\Phi(M)$.

We call this a small model property.

Example

Let $\Phi(M)$ state that ω_1^M is countable in V and M has uncountable height. S_Φ then states that ω_1^V is inaccessible in $L(x)$ for every real x.

Solovay showed this is equivalent to the statement that all Π^1_1 sets have the perfect set property.

Characterisations of PD

We isolated a notion called stable iterability of an operator $\mathbb{M}$.

It says essentially that for an iteration tree T on some $\mathbb{M}(x)$ of limit length, one find the right branch in $\mathbb{M}(T)$, and any of its generic extensions by small forcing, in a uniform way.

We derived the following characterisation from well-known results.

Corollary

The following statements are equivalent for any n:

1. *Π^1_{n+1}-determinacy.*
2. *There exists an n-tall stably $\mathbb{M}$-iterable operator $\mathbb{M}$.*

- For example, the $\mathbb{M}_n^{\#}$-operator from inner model theory is stably $\mathbb{M}_k^{\#}$-iterable for any $k \geq n - 1$.

Preservation of PD

The notion of stable iterability is defined so that one can extend the operator to generic extensions.

Proposition (Schilhan, Schürz, S.)

Suppose that $\mathbb{M}$ is any stably $\mathbb{M}$-iterable $\mathbb{P}$-amenable operator and $\mathbb{P}$ is captured over $\mathbb{M}$.

1. In any $\mathbb{P}$-generic extension, $\mathbb{M}$ can be extended to an $\mathbb{M}^*$-iterable operator $\mathbb{M}^*$.
2. If $\mathbb{M}$ is n-tall, then $\mathbb{M}^*$ is n-tall.

It follows that any proper-for-candidates Suslin forcing preserves PD level-by-level.

Preservation of PD

Corollary

Any Σ^1_3 *proper-for-candidates forcing preserves* Σ^1_2*-determinacy.*

Uses that $M_1^{\#}(x)$ is amenable for Σ^1_3 sets.

- This contrasts David's result that Σ^1_3 c.c.c. forcings can destroy analytic determinacy.

Axiom A

Lemma (Schilhan, Schürz, S.)

Suppose that ω_1 is inaccessible to the reals, $\mathbb{P} = (P, \vec{\lesssim})$ satisfies Axiom A, P is a set of reals and the following conditions hold for any real x:

1. *$\mathbb{P} \cap L[x] \in L[x]$ satisfies Axiom A in $L[x]$.*
2. *For countable subsets $A \in L[x]$ of $\mathbb{P}$ and $p \in \mathbb{P}$, the statements*

   "A is an antichain"

   "A is predense below p"

   are absolute between $L[x]$ and V.

Then $\mathbb{P}$ captures itself.

Given 1. and 2. for all transitive models M instead of $L[x]$, $\mathbb{P}$ is proper-for-candidates.

Axiom A

Proof.

Let $\dot{G}$ be a $\mathbb{P}$-name for the $\mathbb{P}$-generic filter and $\tau \in H_{\omega_1}$ a name for a real, coded by a real x. We claim that

$$D = \{q \in \mathbb{P} \colon \exists p \geq q \ \ q \Vdash_{\mathbb{P}} \dot{G} \cap L[x,p] \text{ is } \mathbb{P}\text{-generic over } L[x,p]\}$$

is dense.

Fix $p \in \mathbb{P}$. Let $\langle A_n \mid n \in \omega \rangle$ enumerate all maximal antichains $A \in L[x,p]$ of $\mathbb{P} \cap L[x,p]$ in $L[x,p]$.

- Using Axiom A in $L[x,p]$, construct a fusion sequence $\langle p_n \mid n \in \omega \rangle$ below p with $p_n \in L[x,p]$ such that $L[x,p] \models \{p' \in A_n \colon \ p' \parallel p_n\}$ is countable.
- Using Axiom A in V, find a condition $q \in \mathbb{P}$ with $q \leq p_n$ for all $n \in \omega$.

To see that q is as required, let $A \in L[x,p]$ be a maximal antichain in $\mathbb{P} \cap L[x,p]$. Then $A = A_n$ for some $n \in \omega$.

Then $\{p' \in A_n \colon p' \parallel p_n\}$ is maximal below p_n in V by the absoluteness assumption.

Since $q \leq_{\mathbb{P}} p_n$, $q \Vdash_{\mathbb{P}} \dot{G} \cap A_n \neq \emptyset$. □

Background

Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha \colon \alpha < \kappa \rangle$ is a countable support iteration of proper-for-candidates strongly Suslin forcings of countable length α.

Suppose that M is a transitive model containing $\mathbb{P}$.

- Judah and Shelah defined a procedure to map a $\mathbb{P}$-generic filter G over V to a filter G^M on $\mathbb{P}^M$.

Theorem (Judah, Shelah 1988)

G^M is $\mathbb{P}^M$-generic over M.

- Similar results (Judah, Shelah and Goldstern) for iterations of uncountable length
- Their main application: force the Borel conjecture over Solovay's model while preserving the Baire property for projective sets.
- Technique used in work of Goldstern, Kellner and others

The proof works for models $L[x]$ if ω_1 is inaccessible to the reals.

Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha < \kappa \rangle$ is a countable support iteration of proper-for-candidates strongly Suslin forcings of length $\geq \omega_2$.

Problem

Does every real in a $\mathbb{P}$-generic extension $V[G]$ have a nice name $\sigma = \{(\check{n}, p) \mid p \in A_n\}$ such that for each $n \in \omega$,

$$\bigcup_{p \in A_n} \mathrm{supp}(p_n)$$

is countable?

The following arguments find a way around this.

Results

Theorem (Schilhan, Schürz, S.)

Suppose that ω_1 is inaccessible in $L(x)$ for every real x. Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha \colon \alpha < \kappa \rangle$ is a countable support iteration of proper-for-candidates Suslin forcings such that for every $\alpha < \kappa$,

> $\Vdash_{\mathbb{P}_\alpha}$ $\dot{\mathbb{P}}_\alpha$ *is proper-for-candidates in every small generic extension of any $L(x)$.*

Then $\mathbb{P}$ is captured over L by a countable support iteration $\mathbb{Q}$ of proper-for-candidates Suslin forcings of countable length.

Example

If $\mathbb{P}$ is an iteration of Sacks forcing, then $\mathbb{Q}$ is an iteration of Sacks forcing as well.

- Therefore, every real in $V[G]$ is contained in a generic extension of some $L(x)$.
- The above lifting argument shows that $\forall x\, x^{\#}$ exists is preserved.

Corollary (Schilhan, Schürz, S.)

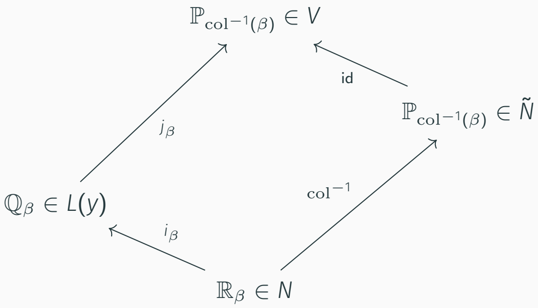
The combination of the Borel conjecture and analytic determinacy is consistent, if latter is consistent.

Proof sketch

Let

- $p \in \mathbb{P}$, τ a $\mathbb{P}$-name for a real and $\tilde{N} \prec H_\theta$ countable with $p, \mathbb{P}, \tau \in \tilde{N}$.
- $\mathrm{col}\colon \tilde{N} \to N$ be the Mostowski collapse and $\alpha := \mathrm{col}(\kappa)$.
- y a real coding N.

In N, $\mathrm{col}(\mathbb{P})$ is a countable support iteration $\mathbb{R} = \mathbb{R}_\alpha = \langle \mathbb{R}_\beta, \dot{\mathbb{R}}_\beta : \beta < \alpha \rangle$ of Suslin forcings.



Proof sketch

For each $\beta \leq \alpha$, we define

- a countable support iteration of Suslin forcings $\mathbb{Q}_\beta = \langle \mathbb{Q}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \beta \rangle$ in $L(y)$,
- a map $i_\beta : \mathbb{R}_\beta \to \mathbb{Q}_\beta$ in $L(y)$ and
- a map $j_\beta : \mathbb{Q}_\beta \to \mathbb{P}_{\mathrm{col}^{-1}(\beta)}$ in V.

We say that a $\mathbb{P}_{\mathrm{col}^{-1}(\beta)}$-generic filter G over V is

- $(L(y), \mathbb{Q}_\beta)$-generic if $j_\beta^{-1}(G)$ is a $\mathbb{Q}_\beta$-generic filter over $L(y)$
- $(N, \mathbb{R}_\beta)$-generic whenever $i_\beta^{-1}(j_\beta^{-1}(G))$ is an $\mathbb{R}_\beta$-generic filter over N.

Similarly, we say that a $\mathbb{Q}_\beta$-generic filter H over $L(y)$ is

- $(N, \mathbb{R}_\beta)$-generic whenever $i_\beta^{-1}(H)$ is an $\mathbb{R}_\beta$-generic filter over N.

We will ensure that for each $\beta \leq \alpha$:

1. $i_\xi(r \upharpoonright \xi) = i_\beta(r) \upharpoonright \xi$ for every $\xi < \beta$, $r \in \mathbb{R}_\beta$,
2. $j_\xi(q \upharpoonright \xi) = j_\beta(q) \upharpoonright \mathrm{col}^{-1}(\xi)$ for every $\xi < \beta$, $q \in \mathbb{Q}_\beta$.
3. $i_\beta(r_0) \leq_{\mathbb{Q}_\beta} i_\beta(r_1)$ if $r_0 \leq_{\mathbb{R}_\beta} r_1$ for all $r_0, r_1 \in \mathbb{R}_\beta$,
4. $j_\beta(q_0) \leq_{\mathbb{P}_{\mathrm{col}^{-1}(\beta)}} j_\beta(q_1)$ if $q_0 \leq_{\mathbb{Q}_\beta} q_1$ for all $q_0, q_1 \in \mathbb{Q}_\beta$,
5. for any $\mathbb{P}_{\mathrm{col}^{-1}(\beta)}$-generic G over V such that G is $(L(y), \mathbb{Q}_\beta)$-generic and $(N, \mathbb{R}_\beta)$-generic, for every $r \in \mathbb{R}_\beta$:
$$\mathrm{col}^{-1}(r) \in G \Leftrightarrow j_\alpha(i_\alpha(r)) \in G$$

Proof sketch

Remark

How do we get the forcings $\dot{\mathbb{Q}}_\alpha$ right? Suppose that

- G is $\mathbb{P}_{\pi(\beta)}$-generic over V,
- G is $(M(y), \mathbb{Q}_\beta)$-generic and
- G is $(N, \mathbb{R}_\beta)$-generic.

By 5., for any $\mathbb{R}_\beta$-name for a real $\dot{x} \in N$,

$$\dot{x}^{i_\beta^{-1}(j_\beta^{-1}(G))} = \dot{x}^{\mathrm{col}(G)} = \mathrm{col}^{-1}(\dot{x})^G.$$

Claim

For every $q \in \mathbb{Q}$ there exists some $p \leq_{\mathbb{P}} j(q)$ in $\mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} \text{“}\dot{G} \text{ is } \mathbb{Q}\text{-generic over } L(y)\text{”}.$$

- The proof is similar to the preservation of properness.
- Use this to show that in $L(y)$, $\mathbb{Q}$ is an iteration of proper-for-candidates forcings.

Similar arguments show that $j_\alpha^{-1}[G]$ is $\mathbb{Q}$-generic over $L(y)$ as required.

Results

We further obtained general results about not adding new classes to thin equivalence relations, building on work of Hjorth.

- The proofs use transitivity of the relation.
- Relaxing this requirement leads to thin graphs.

Theorem (Schilhan, Schürz, S.)
Suppose that analytic determinacy holds and G is an absolutely Δ^1_3 *thin graph. After forcing with a countable support iteration of Sacks forcing, any new real has an edge to some ground model real.*

Regularity properties

Theorem (follows from Pawlikowski 1986, Judah, Shelah 1989)

1. *Cohen forcing preserves the statement: all $\mathbf{\Delta}^1_2$ sets have the Baire property.*
2. *Random forcing preserves the statement: all $\mathbf{\Delta}^1_2$ sets are Lebesgue measurable.*

Theorem (Judah, Shelah 1988, Goldstern, Shelah 1992)

Over Solovay models, countable support iterations of Suslin proper-for-candidates forcings preserve the property of Baire of all projective sets.

Results

Theorem (Schilhan, Schürz, S.)
If $\mathbb{P}$ satisfies a uniform version of capturing for Cohen forcing, then $\mathbb{P}$ preserves the statement:

Every Δ^1_2 set has the Baire property.

For example: any countable support product or iteration of Sacks forcing works.

Future directions

It is natural to aim for similar preservation results for stronger determinacy principles.

Question

Does the above class of forcings preserve determinacy in $L(\mathbb{R})$?

We would like to see that simple definability of the forcings is necessary.

Question

Can Mathias forcing with an ultrafilter destroy analytic determinacy?

Future directions

Ishiu (2005) proved that $<\omega_1$-proper is equivalent to Axiom A.

Question

Is there a similar result for the analogue for $<\omega_1$-proper for proper-for-candidates?

The fine line between proper and proper-for-candidates deserves to be better understood.

Question

Does proper imply proper-for-candidates for Borel forcings, assuming analytic or projective determinacy?

This is closely related to a question of Ikegami (2010) on preservation of analytic determinacy by simply definable proper forcings.