Dichotomies for open dihypergraphs on generalized Baire spaces

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- Philipp Schlicht, Dorottya Sziráki:
  The open dihypergraph dichotomy for generalized Baire spaces,
  80 pages, in preparation

Motivation: Ramsey theory

It is natural to wonder whether Ramsey's theorem for n-tuples of natural numbers can be extended to the set of real numbers.

- Sierpinski's counterexample 1937: a partition of pairs of reals in two pieces with no uncountable homogeneous set
- Galvin 1968: Ramsey's theorem for open graphs on the reals
- Blass 1981: A generalization to Borel n-hypergraphs on the reals

Galvin's theorem can be strengthened.

- Todorcevic's open graph axiom 1989
- Feng's open graph dichotomy for analytic sets 1993

Feng's theorem implies one of the most basic descriptive set theoretic dichotomies: the perfect set property.

The graph-theoretic approach

In recent years, graph dichotomies provided new proofs of old and new theorems in descriptive set theory.

Kechris, Solecki, Todorcevic and Miller proved results for analytic graphs (variants of the G_0-dichotomy) that imply:

- Suslin's perfect set property of analytic sets
- Lusin and Novikov's uniformization of Borel sets with countable sections
- Feng's open graph dichtomy
- Silver's theorem on coanalytic equivalence relations

The open graph dichotomy

A graph G is a symmetric relation with no loops.

A graph G on a space X is an open graph if it is an open subset of $X \times X$ without the diagonal.

Definition (Feng 1993)

$\mathrm{OGD}_\omega(X)$ states that for any open graph G on X, either

1. G has an ω-coloring or
2. G has a perfect complete subgraph.

G has an ω-coloring if and only if X is the union of countably many G-independent sets.

A high-dimensional dichotomy

Carroy, Miller and Soukup 2020 found an infinite dimensional version of Feng's open graph dichotomy.

Note the following restrictions:

- Farah, Todorcevic 1995: The open graph dichotomy fails for closed graphs.
- Farah, Todorcevic 1995, He 2005: The open 3-hypergraph dichotomy fails.

One thus has to consider directed hypergraphs.

A κ-dihypergraph on X is a set of nonconstant sequences in ${}^{\kappa}X$.

A high-dimensional dichotomy

We fix the box topology on ${}^{\omega}X$ with basic open sets $\prod_{i<\omega} U_i$, where each U_i is open in X.

Definition (Carroy, Miller, Soukup 2020)

$\mathrm{ODD}^{\omega}_{\omega}(X)$ states that for any box-open ω-dihypergraph H on X, either

1. H has a ω-coloring or
2. there is a continuous homomorphism $f : {}^{\omega}\omega \to X$ from $\mathbb{H}_{\omega_\omega}$ to H.

$$\mathbb{H}_{\omega_\omega} = \left\{\vec{x} \in {}^{\omega}({}^{\omega}\omega) \mid \exists t \in {}^{<\omega}\omega \ \ \forall n \in \omega \ t^\frown\langle n\rangle \subseteq x_n\right\}$$

$\mathrm{ODD}^{\omega}_{\omega}(X, H)$ states that this holds for H.

[Drawing of hypergraphs]

Example

$\mathrm{ODD}^2_\omega(X)$ implies $\mathrm{OGD}_\omega(X)$.

To see this, suppose $i < \omega$ is least with $x(i) \neq y(i)$. Let

$$\langle x, y \rangle \in \mathbb{H}_{\omega_2} \iff x(i) = 0 \wedge y(i) = 1.$$

Then the complete graph $\mathbb{K}_{\omega_2}$ on ${}^\omega 2$ is the smallest (symmetric) graph containing $\mathbb{H}_{\omega_2}$.
Thus a continuous homomorphism $f : {}^\omega 2 \to X$ from $\mathbb{H}_{\omega_2}$ to a graph G is also a homomorphism from $\mathbb{K}_{\omega_2}$ to G.

Note that f is injective. So G has a perfect complete subgraph.

Applications

Theorem (CMS 2020)

$\mathrm{ODD}^{\omega}_{\omega}(X)$ *holds for all analytic subsets* X *of* ${}^{\omega}\omega$.
It holds for all subsets X, *assuming* AD.

They prove a number of applications:

1. The Hurewicz dichotomy for X: either
    - X is contained in a K_σ set, or
    - X contains a closed subset homeomorphic to ${}^{\omega}\omega$.
2. The Kechris-Louveau-Woodin dichotomy for X: For any set Y that is disjoint from X, there exists either:
    - An F_σ set separating X from Y, or
    - A Cantor set $C \subseteq X \cup Y$ such that
        - $X \cap C$ is homeo. to ${}^{\omega}\omega$ and
        - $Y \cap C$ is homeo. to $\mathbb{Q}$.

3. The Jayne-Rogers theorem on piecewise continuous functions with closed pieces on X.
4. A theorem of Lecomte and Zeleny on $\mathbf{\Delta}^0_2$-measurable ω-colorings on X.

Generalised Baire spaces

κ always denotes an uncountable cardinal with $\kappa^{<\kappa} = \kappa$.
Definitions are analogous:

- The κ-Baire space ${}^\kappa\kappa$ is the set of functions $x : \kappa \to \kappa$ with the bounded topology. The basic open sets are

$$N_t = \{x \in {}^\kappa\kappa \mid t \subseteq x\}$$

  for all $t \in {}^{<\kappa}\kappa$.
- The κ-Cantor space ${}^\kappa 2$ has subspace topology.
- κ-Borel sets are generated from open sets by closing under unions and intersections of size κ and negations.
- κ-analytic sets are continuous images of closed sets.

From PSP to OGD

Relative to an inaccessible cardinal:

Theorem (Lücke, Motto Ros, S. 2016)

The topological Hurewicz dichotomy for all κ-analytic subsets of ${}^{\kappa}\kappa$ is consistent.

Theorem (S. 2017)

The perfect set property (PSP) for all definable subsets of ${}^{\kappa}\kappa$ is consistent.

Theorem (Sziraki 2018)

The open graph dichotomy (OGD) for all κ-analytic subsets of ${}^{\kappa}\kappa$ is consistent.

By definable we mean definable from a sequence in ${}^{\kappa}\mathrm{Ord}$.

Definition

$\mathrm{ODD}^{\kappa}_{\kappa}(X)$ states that for any box-open κ-dihypergraph H on X, either

1. H has a κ-coloring or
2. there is a continuous homomorphism $f : {}^{\kappa}\kappa \to X$ from $\mathbb{H}_{^\kappa\kappa}$ to H.

$$\mathbb{H}_{^\kappa\kappa} = \left\{ \vec{x} \in {}^{\kappa}({}^{\kappa}\kappa) \mid \exists t \in {}^{<\kappa}\kappa \;\; \forall i \in \kappa \; t^\frown\langle i\rangle \subseteq x_i \right\}$$

$\mathrm{ODD}^{\kappa}_{\kappa}(X, H)$ states that this holds for H.
$\mathrm{ODD}^{\alpha}_{\kappa}$ denotes the version for α-dihypergraphs.

Theorem (Sziraki, S. 2021)
After a Levy collapse of λ to κ^+, $\mathrm{ODD}_\omega^\omega(X, H)$ holds for all definable subsets X of ${}^\kappa\kappa$ and for all

1. *definable box-open κ-dihypergraphs H on X, if λ is inaccessible in the ground model.*
2. *arbitrary box-open κ-dihypergraphs H on X, if λ is Mahlo in the ground model.*

Applications

Simplified proofs also work for ω instead of κ, so:

- All applications of CMS are consistent relative to an inaccessible or Mahlo cardinal. They do not need AD.

Variants

Consider the following versions of $\mathrm{ODD}^{\kappa}_{\kappa}(X, H)$ with the condition on the homomorphism is strengthened.

- $\mathrm{ODDC}^{\kappa}_{\kappa}(X, H)$: homeomorphism onto a closed image
- $\mathrm{ODDH}^{\kappa}_{\kappa}(X, H)$: homeomorphism onto its image
- $\mathrm{ODDI}^{\kappa}_{\kappa}(X, H)$: injective

Variants

$$\begin{tikzcd}[column sep=large] \mathrm{ODDC}^{\omega}_{\omega}(X,H) \arrow[r, two heads] & \mathrm{ODDH}^{\omega}_{\omega}(X,H) \arrow[r, two heads, dotted, shift left, "?"] & \mathrm{ODDI}^{\omega}_{\omega}(X,H) \arrow[l, dotted, shift left, "?"] \arrow[r, two heads] & \mathrm{ODD}^{\omega}_{\omega}(X,H) \end{tikzcd}$$

The $\kappa = \omega$ case

$$\begin{tikzcd}[column sep=large] \mathrm{ODDC}^{\kappa}_{\kappa}(X,H) \arrow[r, two heads] & \mathrm{ODDH}^{\kappa}_{\kappa}(X,H) \arrow[r, two heads, dashed, shift left, "\Diamond^{i}_{\kappa}"] \arrow[rr, bend left] & \mathrm{ODDI}^{\kappa}_{\kappa}(X,H) \arrow[l, dotted, shift left, "?"] \arrow[r, leftrightarrow, dashed, shift left, "\Diamond^{i}_{\kappa}"] \arrow[r, two heads, dotted, shift right, "?"'] & \mathrm{ODD}^{\kappa}_{\kappa}(X,H) \end{tikzcd}$$

$\kappa = \omega_1$ or κ is weakly inaccessible but not inaccessible

$$\begin{tikzcd}[column sep=large] \mathrm{ODDC}^{\kappa}_{\kappa}(X,H) \arrow[r, two heads] & \mathrm{ODDH}^{\kappa}_{\kappa}(X,H) \arrow[r, two heads] & \mathrm{ODDI}^{\kappa}_{\kappa}(X,H) \arrow[r, leftrightarrow] & \mathrm{ODD}^{\kappa}_{\kappa}(X,H) \end{tikzcd}$$

$\kappa \geq \omega_2$ is a successor cardinal or is inaccessible

The implications from left to right hold by definition.

$A \longleftrightarrow B$: A and B are equivalent for all X, H.

$A \twoheadrightarrow B$: A implies B for all X, H and the implication is strict, i.e., there exist X, H such that the reverse implication fails.

solid arrow: provable for all κ with $\kappa^{<\kappa} = \kappa$.
dashed arrow: consistent and follows from the assumption in the superscript.
dotted arrow: its consistency is an open question.

[Discussion of diagram and $\Diamond^i_{\kappa}$]

Definition

Let $\mathbb{D}_\kappa$ denote the κ-dimensional box-open dihypergraph on ${}^\kappa\kappa$ consisting of all non-constant sequences $\langle x_\alpha : \alpha < \kappa \rangle$ which are dense in some basic open subset of ${}^\kappa\kappa$.[1]

Lemma

$\mathrm{ODD}^\kappa_\kappa(X, \mathbb{D}_\kappa)$ *holds, but* $\mathrm{ODDH}^\kappa_\kappa(X, \mathbb{D}_\kappa)$ *fails.*

[1] I.e., $\{x_\alpha : \alpha < \kappa\} \cap N_t$ is a dense subset of N_t for some $t \in {}^{<\kappa}\kappa$.

Notation: Let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic, where $\lambda > \kappa$ is inaccessible. For each $\alpha < \lambda$, let $G_\alpha = G \cap \mathrm{Col}(\kappa, {<}\alpha)$. Write

$$X_{\varphi,a} = \{x \in {}^\kappa\kappa : \varphi(x, a)\}$$

Lemma

Suppose $X \subseteq {}^\kappa\kappa$. If X is definable in $V[G]$ or λ is Mahlo in V, then

$$X \cap V[G_\nu] \in V[G_\nu]$$

for stationarily many $\nu < \lambda$.

Proof sketch.

If X is definable in $V[G]$, the claim holds for a tail of $\nu < \kappa$, since the tail forcings are homogeneous.

Now suppose that λ is Mahlo in V.

Step 1: Reflection

Let $\dot{X}$ be a name for X. Define $f : \lambda \to \lambda$ as follows.

For $\alpha < \lambda$ and a nice $\mathrm{Col}(\kappa, {<}\alpha)$-name $\dot{x} \in V$ for a subset of $\kappa \times \kappa$, let $A_{\dot{x}}$ be a maximal antichain in $\mathrm{Col}(\kappa, {<}\lambda)$ deciding $\dot{x} \in \dot{X}$.

Since $\mathrm{Col}(\kappa, {<}\lambda)$ has the λ-c.c., let $f(\alpha) < \lambda$ be such that $A_{\dot{x}} \subseteq \mathrm{Col}(\kappa, {<}f(\alpha))$ for all such nice names $\dot{x}$.

The set S of inaccessible closure points of f is stationary, since λ is Mahlo.

Claim

$X \cap V[G_\nu] \in V[G_\nu]$ for all $\nu \in S$.

Let

$$F_\nu(\dot{x}^{G_\nu}) = \begin{cases} 1 & \text{if } p \Vdash^V_{\mathrm{Col}(\kappa,<\lambda)} \dot{x} \in \dot{X} \text{ for some } p \in G_\nu, \\ 0 & \text{if } p \Vdash^V_{\mathrm{Col}(\kappa,<\lambda)} \dot{x} \notin \dot{X} \text{ for some } p \in G_\nu. \end{cases}$$

F_ν is the characteristic function of $X \cap V[G_\nu]$, since $G_\nu \subseteq G$.

Step 2: Independent trees

In $V[G]$, suppose $a \in {}^\kappa\mathrm{Ord}$. Write

$$X_{\varphi,a} = \{x \in {}^\kappa\kappa \mid \varphi(x,a)\}.$$

$$\mathcal{T}^{\mathrm{ind}} = \{T \subseteq {}^{<\kappa}\kappa \mid T \text{ is a tree, } [T] \text{ is } R\text{-independent}\}.$$

Then $\mathcal{T}^{\mathrm{ind}} \cap V[G_\nu] \in V[G_\nu]$ for some $\nu < \lambda$ with $a \in V[G_\nu]$ by the previous step. We can assume $V[G_\nu] = V$.

If R has no κ-coloring, then for some $\gamma < \lambda$:

$$(X_{\varphi,a} \setminus \bigcup\{[T] \mid T \in \mathcal{T}_V^{\mathrm{ind}}\}) \cap V[G_\gamma] \neq \emptyset.$$

Step 2: Independent trees

In V, let $\dot{x}$ be a $\mathrm{Col}(\kappa, {<}\gamma)$-name for an element of $X_{\varphi,a}$ such that $1_{\mathrm{Col}(\kappa,<\gamma)} \Vdash \dot{x} \notin [T]$ for all $T \in \mathcal{T}_V^{\mathrm{ind}}$. For any $p \in \mathrm{Col}(\kappa, {<}\gamma)$, let

$$T^{\dot{x},p} = \{t \in {}^{<\kappa}\kappa \mid \exists q \leq p \ \ q \Vdash t \subseteq \dot{x}\}$$

denote the tree of possible values for $\dot{x}$ below p.

Lemma

1. $1_{\mathrm{Col}(\kappa,<\gamma)} \Vdash$ "*$\dot{x} \in X_{\varphi,a}$ in every further $\mathrm{Col}(\kappa, {<}\lambda)$-gen. extension.*"
2. $T^{\dot{x},p} \notin \mathcal{T}_V^{\mathrm{ind}}$ *for all $p \in \mathrm{Col}(\kappa, {<}\gamma)$.*

Proof of 2. $\quad p \Vdash \dot{x} \in [T^{\dot{x},p}]$. □

We now assume $\dot{x}$ is an $\mathrm{Add}(\kappa, 1)$-name.

Step 3: Construction of a forcing

The forcing will construct the required homomorphism. The point is to avoid subsets of κ with bad quotients.

We construct a forcing $\mathbb{Q}$ such that:

1. $\mathbb{Q}$ is equivalent to $\mathrm{Add}(\kappa, 1)$.
2. Suppose that $V[H]$ is any $\mathbb{Q}$-generic extension of V. $\mathbb{Q}$ adds a map $g : ({}^{\kappa}\kappa)^{V[H]} \to ({}^{\kappa}\kappa)^{V[H]}$ such that for each $y \in ({}^{\kappa}\kappa)^{V[H]}$,
   - $g(y)$ is $\mathrm{Add}(\kappa, 1)$-generic over V,
   - $V[H]$ is a $\mathrm{Add}(\kappa, 1)$-generic extension of $V[g(y)]$, and
   - $\dot{x}^{g(y)} \in X_{\varphi,a}$.

   $f : {}^{\kappa}\kappa \to X$, $f(y) = \dot{x}^{g(y)}$ is continuous.
3. f is a homomorphism from $\mathbb{H}_{{}^{\kappa}\kappa}$ to R.

The main work is to prove properties of $\mathbb{Q}$.

Applications

Many of the applications of $\mathrm{ODD}_\kappa^\kappa$ use arguments of Carroy, Miller and Soukup 2020.

Compactness of 2^ω needs to be avoided.

Applications: Hurewicz dichotomy

A subset X of ${}^{\kappa}\kappa$ is called

- X is κ-compact if every open cover of X has a subcover of size $<\kappa$.
- X is K_κ if X is the union of κ many κ-compact sets.

Note that 2^κ is κ-compact if and only if κ is weakly compact (folklore).

Definition

The topological Hurewicz dichotomy $\mathrm{THD}_\kappa(X)$ for X states that either

- X is contained in a K_κ subset of ${}^{\kappa}\kappa$, or
- X contains a closed subset of ${}^{\kappa}\kappa$ that is homeomorphic to ${}^{\kappa}\kappa$.

Applications: Hurewicz dichotomy

For a subset X of ${}^\kappa\kappa$, let H_X denote the κ-dihypergraph on X of all injective sequences in X with no convergent subsequence.

Proposition (Sziraki, S.)

1. There is an κ-coloring of $H_X \restriction Y$ iff Y is contained in a K_κ set.
2. X contains a closed subset of ${}^\kappa\kappa$ that is homeomorphic to ${}^\kappa\kappa$ iff there exists a continuous homomorphism from $\mathbb{H}_{{}^\kappa\kappa}$ to X.

Thus $\mathrm{THD}_\kappa(X)$ is equivalent to $\mathrm{ODD}_\kappa^\kappa(X, H_X)$.

[Sketch of proof]

Applications: Hurewicz dichotomy

Suppose that T is a subtree of ${}^{<\kappa}\kappa$ and X is a subset of ${}^{\kappa}\kappa$.

- T is a superperfect tree if T is $<\kappa$-closed and every node t of T has a κ-splitting node above it.
- X is a superperfect set if $X = [T]$ for a superperfect tree T.
- T is $<\kappa$-splitting if every node of T has $<\kappa$ many direct successors.

Definition

The Hurewicz dichotomy ${}_{\kappa}(X)$ states that either

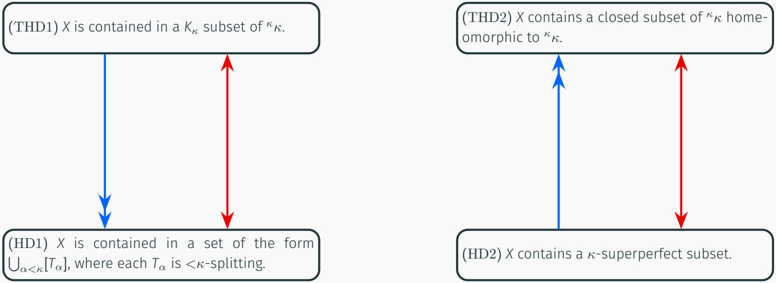
- X has a superperfect subset, or
- $X \subseteq \bigcup_{\alpha<\kappa}[T_\alpha]$ for some $<\kappa$-splitting subtrees T_α of ${}^{<\kappa}\kappa$.

Applications: Hurewicz dichotomy

Proposition (Sziraki, S.)

$\mathrm{HD}_\kappa(X)$ is equivalent to $\mathrm{ODD}^\kappa_\kappa(X, H'_X)$. for some definable box-open κ-dihypergraph on X.

Applications: Hurewicz dichotomy



A $\twoheadrightarrow$ B: A implies B for all X, and the reverse implication fails for $X = {}^\kappa 2$ when $\kappa > \omega$ **is not weakly compact**.

A $\longleftrightarrow$ B: A and B are equivalent for all X when κ **is weakly compact or** $\kappa = \omega$.

Theorem (Sziraki, S. 2022)
$\mathrm{ODD}_{\kappa}^{\kappa}(X)$ *implies a version of the Kechris-Louveau-Woodin dichotomy for subsets X of ${}^{\kappa}\kappa$.*

Väänänen's perfect set game

Definition

Väänänen's perfect set game $\mathcal{V}_\xi(X)$ of length $\xi \leq \kappa$ for a subset X of ${}^\kappa\kappa$ is played by two players I and II as follows. I plays a strictly increasing continuous sequence $\langle \gamma_\alpha : \alpha < \xi \rangle$ of ordinals below κ with $\gamma_0 = 0$. II plays an injective sequence $\langle x_\alpha : \alpha < \xi \rangle$ of elements of X with $x_\alpha \supseteq (x_\beta \restriction \gamma_{\beta+1})$ for all $\beta < \alpha$.

I	γ_0		γ_1		$\dots$		γ_α		$\dots$	
II		x_0		x_1		$\dots$		x_α		$\dots$

Each player loses immediately if they do not follow these requirements. If both follow the rules in all rounds, then II wins.

Moreover, the game $\mathcal{V}_\xi(X, x)$ (this is what Väänänen defined) is defined just like $\mathcal{V}_\xi(X)$ for any x in X, except that II must play $x_0 = x$ in the first round.

Theorem (Sziraki, S. 2022)
$\mathrm{ODD}_\kappa^\kappa(X)$ *implies that either*

- $|X| \leq \kappa$ *and I wins* $\mathcal{V}_\kappa(X)$*, or*
- *II wins* $\mathcal{V}_\kappa(X)$*.*

in particular, $\mathcal{V}_\kappa(X)$ *is determined.*

Väänänen proved the consistency of a weaker version of this statement from a measurable cardinal in 1993.

[Discussion of asymmetric Baire property]

Separating the variants

Can the above dichotomies be separated for different dimensions?
For instance, is it consistent that $\mathrm{OGD}_\kappa(X)$ holds, but $\mathrm{ODD}_\kappa^\kappa(X)$ fails?
Note that all models above are Levy collapses.

Inaccessibles are necessary for the above results.
Mahlo cardinals are needed for the proofs, but are they necessary for the results?
This would separate the variant for arbitrary dihypergraphs from the definable variant.

Separating the variants

To separate ODD from the version with injective homomorphisms at ω_1, one can consider models where CH holds, but $\diamondsuit_{\omega_1}$ fails.

Jensen's and Jonsbraten constructed such a model for the Suslin problem with CH.

A simpler construction of such a model is possible using recent work of Aspero and Mota.

Future directions

Further applications are known in the countable case.
We hope to generalise some of them.

ODD, PSP and some versions of the Hurewicz dichotomy at κ can be formulated as the determinacy of a game of length κ.
We aim to find natural class of determined games that include the above.