Baby measurable cardinals

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Based on work in progress with Victoria Gitman.

- Victoria Gitman, Philipp Schlicht: Baby measurable cardinals
  29 pages, in preparation

Motivation

- The area of Ramsey-like cardinals aims to understand large cardinals between weakly compacts and measurables.
- We study n-baby measurable cardinals, introduced by Bovykin and McKenzie to answer a question of Holmes.

Ramsey-like cardinals

Definition: A cardinal κ is Ramsey if every coloring $f : [\kappa]^{<\omega} \to 2$ has a homogeneous set of size κ.

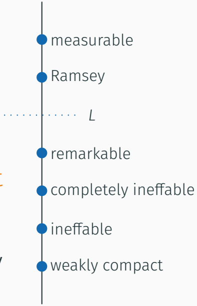
Definition: A cardinal κ is remarkable if it is virtually Magidor supercompact.

Definition: A cardinal κ is completely ineffable if there exists a collection of subsets of κ that is closed under applications of Ramsey's theorem for pairs.

Definition: A cardinal κ is ineffable if for every sequence $\{A_\xi \mid \xi < \kappa\}$ with $A_\xi \subseteq \xi$, there is a $A \subseteq \kappa$ and a stationary set S such that for all $\xi \in S$, $A \cap \xi = A_\xi$.

Theorem: (Kunen, Jensen) A cardinal κ is ineffable if and only if every coloring $f : [\kappa]^2 \to 2$ of pairs of elements of κ in 2 colors has a stationary homogeneous set.

Definition: A cardinal κ is called weakly compact if every coloring $f : [\kappa]^2 \to 2$ has a homogeneous set of size κ.



We always assume that κ is inaccessible.

Definition

- A weak κ-model is a transitive set $M \models \mathrm{ZFC}^-$ of size κ with $H_\kappa \in M$.
- A κ-model is a weak κ-model with $M^{<\kappa} \subseteq M$.
- An basic κ-model is a (not necessarily transitive) set $M \models \mathrm{ZFC}^-$ of size κ with $H_\kappa \in M$ and $M \prec_{\Sigma_0} V$.
- An basic $<\kappa$-closed κ-model is a basic κ-model with $M^{<\kappa} \subseteq M$.

In each case, we will say that M is simple if κ is the largest cardinal in M.[1]

[1] The definition of (weak) κ-models in the literature is slightly different, in particular it is usually not assumed that κ is inaccessible and $H_\kappa \in M$.

(Normal) ultrafilters

Suppose that M is a weak κ-model.

- An M-ultrafilter is a set $U \subseteq P^M(\kappa)$ that is a uniform filter on κ, i.e. it contains the tail sets $\kappa \setminus \alpha$, such that

$$\langle M, \in, U \rangle \models \text{"}U \text{ is a normal ultrafilter on } \kappa.\text{"}$$

An M-ultrafilter U is called good if the ultrapower of M by U is well-founded.

Note: separation and collection may fail badly in the structure $\langle M, \in, U \rangle$.

Weakly amenable ultrafilters

Suppose that M is a weak κ-model, U is an M-ultrafilter, and $j_U : M \to N$ is the ultrapower embedding.

For iterated ultrapowers, we need to define "$j_U(U)$". This works if U is weakly amenable.

Definition

- An M-ultrafilter U is weakly amenable (to M) if for every $A \in M$ with $|A|^M \leq \kappa$, $U \cap A \in M$.
- Amenable means this holds for all $A \in M$.

Weakly amenable M-ultrafilters U are "partially internal" to M.

Proposition: If M is simple, then U is (weakly) amenable to M if and only if $\langle M, \in, U \rangle$ satisfies Σ_0-separation.

Weakly amenable ultrafilters

Suppose M is a weak κ-model and U is an M-ultrafilter.

Definition: An elementary embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$ is κ-powerset preserving if $P^M(\kappa) = P^N(\kappa)$.

Proposition (folklore):

- If U is good and weakly amenable, then the ultrapower $j_U : M \to N$ is κ-powerset preserving.
  - If M is simple, then $M = H^N_{\kappa^+}$.
- If $j : M \to N$ is κ-powerset preserving, then the M-ultrafilter U generated by κ via j is weakly amenable.

Some embedding characterisations

Proposition (folklore): The following are equivalent for an inaccessible cardinal κ:

1. κ is weakly compact.
2. Every subset A of κ is contained in a transitive model $(M, \in)$ of ZFC^- of size κ such that there exists a countably complete (in V) M-normal ultrafilter U on $P(\kappa)^M$.
3. As in 2., but replacing countable completeness of U by having a wellfounded ultrapower.
4. As in 2., but replacing countable completeness of U by the statement that M is $<\kappa$-closed.
5. Same as any of the above, but replacing a transitive M with $M \prec H_\theta$ for any regular $\theta > \kappa$.

Some embedding characterisations

Theorem (essentially Di Prisco, Zwicker 1980): The following are equivalent for an inaccessible cardinal κ:

1. κ is ineffable.
2. κ has the normal filter property, i.e. as in the above characterisation (2.) of weakly compacts, but additionally the diagonal intersection of U is stationary (in V).

Properties I: well-founded targets

Well-founded ultrapowers are already necessary in the embedding characterisation of weakly compact cardinals.

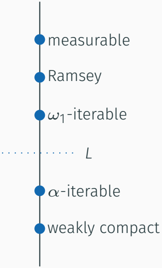
Suppose that M is a weak κ-model.

Definition: An M-ultrafilter U is α-iterable if it is weakly amenable and has α many well-founded iterated ultrapowers. U is iterable if it is α-iterable for every $\alpha \in \mathrm{Ord}$.

Proposition: (Gaifman) If an M-ultrafilter U is ω_1-iterable, then U is iterable.

Theorem: (Kunen) If an M-ultrafilter U is ω_1-complete, then U is iterable.

Definition: (Gitman, Welch) A cardinal κ is α-iterable, for $1 \leq \alpha \leq \omega_1$, if every $A \subseteq \kappa$ is contained in a weak κ-model M that admits an α-iterable M-ultrafilter.



Theorem:

- (Gitman) A 1-iterable cardinal is a limit of ineffable cardinals.
- (Gitman, Welch) A β-iterable cardinal is a limit of α-iterable cardinals for all $\alpha < \beta$.

Properties II: closure

Definition: (Holy, S.) A cardinal κ is α-Ramsey for a regular α with $\omega \leq \alpha \leq \kappa$ if for every $A \subseteq \kappa$ and arbitrarily large regular θ, there is a basic κ-model $M \prec H_\theta$, with $A \in M$, such that $M^{<\alpha} \subseteq M$ for which there is a weakly amenable M-ultrafilter.

Theorem: (Holy, S.)

- A measurable cardinal is a limit of κ-Ramsey cardinals κ.
- An ω_1-Ramsey cardinal is a limit of Ramsey cardinals.
- For $\omega \leq \alpha < \beta \leq \kappa$, a β-Ramsey cardinal κ is a limit of α-Ramsey cardinals.

The proof that the α-Ramsey cardinal form a strict hierarchy is proved via the following game.

Properties II: closure

Definition: (Holy, S.) Fix regular uncountable $\alpha \leq \kappa < \theta$. The game $G^\theta_\alpha(\kappa)$ is played by the challenger and the judge.

At every stage $\gamma < \alpha$:

- the challenger plays an basic $<\kappa$-closed κ-model $M_\gamma \prec H_\theta$ extending his previous moves with $\{\langle M_{\bar\gamma}, \in, U_{\bar\gamma}\rangle \mid \bar\gamma < \gamma\} \in M_\gamma$.
- the judge responds with an M_γ-ultrafilter U_γ extending her previous moves,

The judge wins if she can play for α-many moves and otherwise the challenger wins.

Observations: Suppose the judge wins a run of the game $G^\theta_\alpha(\kappa)$.

- $M = \bigcup_{\gamma<\alpha} M_\gamma$ is closed under $<\alpha$-sequences.
- $U = \bigcup_{\gamma<\alpha} U_\gamma$ is a weakly amenable M-ultrafilter.

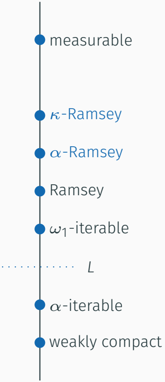
The existence of winning strategies for either player is independent of θ.

Properties II: closure

Theorem: (Holy, S.) The following are equivalent.

- κ is α-Ramsey.
- The challenger does not have a winning strategy in the game $G^{\theta}_{\alpha}(\kappa)$ for some/all θ.

The existence of winning strategies for the judge is connected with ideals with certain closure properties (Foreman, Magidor, Zeman 2020).



Properties III: amenability

Recall that κ is always inaccessible.

Fact: The following are equivalent:

1. A cardinal κ is weakly compact.
2. Every subset A of κ is contained in weak κ-model M that admits a countably complete (in V) M-ultrafilter U on $P(\kappa)^M$.

Theorem (implicit in work of Mitchell):
A cardinal κ is Ramsey if 2. holds as above, but additionally the ultrafilter U is amenable.

Theorem (Holy, Lücke 2021)

The following are equivalent:

1. *A cardinal κ is completely ineffable.*
2. *For all regular $\theta > \kappa$, every subset A of κ is contained in some $M \prec H_\theta$ that admits an M-amenable M-ultrafilter U.*

Properties III: amenability

Let ZFC^-_n denote the restriction to Σ_n-formulas of the axioms and schemes[2] of ZFC^-.

The following definition of n-baby measurable cardinals is a slight (but not equivalent) variant of the definition of n-baby measurables of Bovykin and McKenzie.

Definition:

- A cardinal κ is very weakly n-baby measurable if every subset A of κ is contained in a weak κ-model M that admits an M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_n$.
- A cardinal κ is weakly n-baby measurable if the previous statement holds, but U is additionally good.
- A cardinal κ is n-baby measurable if the previous statement holds, but M is additionally $<\kappa$-closed.

[2] We use the collection instead of the replacement scheme.

Quine's New Foundations NF is a system with unrestricted comprehension for typed formulas. Randall Holmes claimed a proof of its consistency in ZFC a few years ago.

NFU denotes NF with Urelements. This is equiconsistent with a weak subsystem of ZFC by Jensen 1969.

Holmes' 1998 textbook introduced an extension NFUM of NFU to facilitate the formalisation of mathematics. This lead to the original motivation for studying n-baby measurable cardinals.

Ramsey-like cardinals and Kelley-Morse

Definition

- Let KM_U denote KM in the language with a unary hyperclass predicate U and the the assertion:

  "U is a normal ultrafilter on Ord*."*

- Let ZFC^-_U denote ZFC^- in the language with a unary predicate U and the the assertion:

  "U is a normal ultrafilter on the largest cardinal κ*."*

Proposition (Marek?)

KM_U and ZFC^-_U are equiconsistent:

Theorem (Holmes, Solovay 2001)

KM_U and NFUM are equiconsistent:

Theorem (Bovykin, McKenzie 2012)

The following theories are equiconsistent:

1. *ZFC with the scheme of assertions for $n < \omega$:*

   There exists a n-baby measurable cardinal κ such that $V_\kappa \prec_n V$.

2. KM_U

- Bovykin, McKenzie: Ramsey-like cardinals that characterize the exact consistency strength of NFUM
  Preprint, 2012

Proposition (Gitman, S.): A very weakly $n+2$-baby measurable cardinal is an n-baby measurable limit of n-baby measurable cardinals.

To show this, one constructs a κ-model $\bar{M} \in M$ such that $\langle \bar{M}, \in, U \rangle \prec_{\Sigma_{n+1}} \langle M, \in, U \rangle$ and shows $\langle \bar{M}, \in, U \rangle \models \mathrm{ZFC}^-_n$.

n-baby measurables in the hierarchy

Theorem: (Gitman, S.) A weakly 0-baby measurable cardinal below which the GCH holds is a limit of 1-iterable cardinals.

The first step: suppose that

- M is a simple weak κ-model and
- U is a weakly amenable ultrafilter on M.

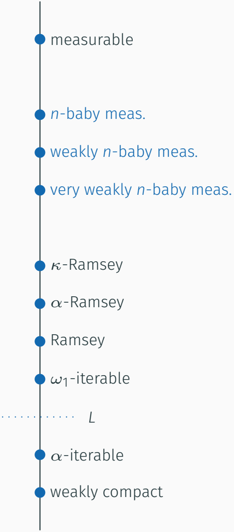
Then $(M, \in, U)$ has a Δ_1-definable global wellorder. If the GCH holds below κ, then its order type is Ord^M.

One can then do Σ_1-recursion along the wellorder.

n-baby measurables in the hierarchy

Theorem: (Gitman, S.) A very weakly 1-baby measurable cardinal is a limit of cardinals α that are α-Ramsey.

We further study stronger versions of n-baby measurable cardinals and characterisations by games similar to that of α-Ramsey cardinals.



Outlook

We aim for some technical improvements, for instance:

Problem
Can the assumption of GCH be avoided in the study of 0*-baby measurable cardinals?*

Is there a sequence of large cardinal notions, unbounded below a measurable cardinal in consistency strength?