Countable ranks at the first and second projective levels

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Based on a joint project with Merlin Carl and Philip Welch.

- Merlin Carl, Philipp Schlicht, Philip Welch:
  Countable ranks at the first and second projective levels
  30 pages, to be submitted

Motivation

A rank is a notion in descriptive set theory that describes natural ranks such as

- The Cantor-Bendixson rank on the set of closed subsets of a Polish space,
- Differentiability ranks on the set of differentiable functions in $C[0,1]$ such as the Kechris-Woodin rank

and many other ranks in descriptive set theory and real analysis.

Examples

1. The natural rank on the Π^1_1 set WO of all wellorders on $\mathbb{N}$ has as αth layer the set WO_α of all wellorders with order type α.

2. The Cantor-Bendixson rank on the Π^1_1 set of countable closed sets C of reals is defined via a sequence of derivatives $C^{(\alpha)}$:

    - $C^{(\alpha+1)}$ is obtained from $C^{(\alpha)}$ by removing isolated points.
    - $C^{(\lambda)} = \bigcap_{\alpha<\lambda} C^{(\alpha)}$ for limit ordinal λ.

    The rank of C is the least α with $C^{(\alpha)} = \emptyset$.

Examples

3. The Kechris-Woodin rank on the Π^1_1 set of differentiable functions $f \in C[0,1]$.

- Let C be a closed subset of $[0,1]$ and $\epsilon > 0$. The derivative $C'_{f,\epsilon}$ removes points where f is close to begin differentiable: $C'_{f,\epsilon}$ consists of all $x \in C$ such that for all $\delta > 0$, there exist rational intervals $[p,q]$ and $[r,s]$ in $B(x,\delta) \cap [0,1]$ such that $[p,q] \cap [r,s] \cap C \neq \emptyset$ and $|\Delta_f(p,q) - \Delta_f(r,s)| \geq \epsilon$.
- The iterated derivatives $D^\alpha_{f,\epsilon}$ are defined by starting from $D^0_{f,\epsilon} = [0,1]$, letting $D^{\alpha+1}_{f,\epsilon} = (D^\alpha_{f,\epsilon})'_{f,\epsilon}$ and forming intersections at limits.

The Kechris-Woodin rank $|f|$ of f is the least ordinal α such that $D^\alpha_{f,\epsilon} = \emptyset$ for all $\epsilon > 0$.

For example, any continuously differentiable function f has rank 1.

Effective descriptive set theory

Effective descriptive set theory studies effectively definable subsets of the Baire space ω^ω:

- A Π^0_1 set is the set $[T]$ of branches through a computable subtree T of $\omega^{<\omega}$.

  These are precisely the sets definable by some Π^0_1-formula $\forall n\ \varphi(x, n)$, where φ is arithmetical.

- A Σ^1_1 set is the projection $p[C]$ of a Π^0_1 subset C of $\omega^\omega \times \omega^\omega$ to the first coordinate.

  These are precisely the sets definable by some Σ^1_1-formula $\exists y\ \varphi(x, y)$, where φ is arithmetical.

- Π^1_1 sets are complements of Σ^1_1 sets.

- Σ^1_2 sets are projections of Π^1_1 sets.

Motivation for ranks

A rank layers a set of reals by representing it as a union of a chain of simpler subsets.

The definition of ranks on the next slide can be explained by considering an infinite time algorithm M that runs on real inputs.

Let A denote the set of reals x such that $M(x)$ halts.

For reals x, y, let $x \sqsubseteq y$ if $M(x)$ halts and $M(y)$ does not halt before $M(x)$. This relation defines a rank on A.

A reason why ranks are useful: Since every Π^1_1 set admits a Π^1_1-rank, the class of Π^1_1 sets has the reduction property. Lusin's separation theorem for Σ^1_1 sets follows.

Ranks

A prewellorder $\leq$ is a wellorder without the requirement of antisymmetry. i.e. it is a wellfounded linear quasiorder.

Let Γ denote a collection of subsets of ω^ω such as Π^1_1 or Σ^1_2.

Let $C_y = \{x \mid (x, y) \in A\}$ denote a section of a subset C of $\omega^\omega \times \omega^\omega$.

Definition

A Γ-rank on a set $A \in \Gamma$ is a prewellorder $\leq$ on A with strict part $<$ such that there exist Γ relations $\sqsubseteq$ and $\sqsubset$ with the following properties:

1. *(Left agreement)* For all $x \in A$:
   - 1.1 $\sqsubseteq_x$ equals $\leq_x$.
   - 1.2 $\sqsubset_x$ equals $<_x$.
2. *(Overspill)* $x \sqsubseteq y$ and $x \sqsubset y$ for all $(x, y) \in A \times (2^\omega \setminus A)$.

Its length is the order type of $\leq$.

Definition

Γ has the rank property if every set in Γ admits a Γ-rank.

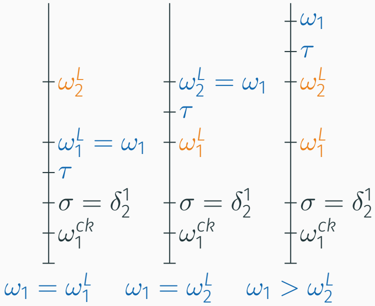
The above examples from descriptive set theory and real analysis are Π^1_1-ranks.

For example, the set WO of wellorders on ω is a Π^1_1 set and the rank on WO with αth layer WO_α is a Π^1_1-rank.
The layers are Borel by the properties of ranks.

Results

Our main result determines the suprema of lengths of countable Π^1_1-ranks and of countable Σ^1_2-ranks.

Let σ and τ denote the suprema of Σ_1- and Σ_2-definable ordinals over L_{ω_1}. Here ω_1 always denotes ω_1^V.



Results

Theorem

The following sets of ordinals all have (strict) supremum τ:

1. 1.1 Π_1-*definable ordinals over* L_{ω_1}
   1.2 Σ_2-*definable ordinals over* L_{ω_1}
2. *Countable ranks of* Σ^1_2 *wellfounded relations*
3. *Lengths of countable*
   3.1 Π^1_1 *ranks*
   3.2 Σ^1_2 *ranks*
4. *Lengths of countable*
   4.1 Σ^1_1 *prewellorders on* Σ^1_1 *sets*
   4.2 *(strict)* Π^1_1 *prewellorders on* Π^1_1 *sets*
   4.3 *strict* Σ^1_2 *prewellorders on* Σ^1_2 *sets*

Results

Using the previous theorem, we show that τ equals the ordinal γ_2^1 studied by Kechris.

We further use it to give short proofs of some results of Kechris, Marker and Sami.

- Kechris, Marker, Sami: Π_1^1 Borel sets
  Journal of Symbolic Logic, 1989

Results

The next diagram summarises what we proved about the suprema of various classes of countable prewellorders.

Let ω_1^{ck} denote the supremum of computable ordinals.
For the rightmost column, assume that $0^{\#}$ exists and let ι_0 denote the first Silver indiscernible.

	ω_1^{ck}	τ	$>\iota_0$
$\mathbf{\Delta}^1_1$	*(strict)* pwo's on 2^ω / $\mathbf{\Delta}^1_1$ sets		
$\mathbf{\Sigma}^1_1$	*strict* pwo's on 2^ω / $\mathbf{\Sigma}^1_1$ sets	pwo's on 2^ω / $\mathbf{\Sigma}^1_1$ sets	
$\mathbf{\Pi}^1_1$	pwo's on 2^ω	pwo's on $\mathbf{\Pi}^1_1$ sets	
		strict pwo's on 2^ω / $\mathbf{\Pi}^1_1$ sets	
$\mathbf{\Delta}^1_2$		*(strict)* pwo's on 2^ω / $\mathbf{\Delta}^1_2$ sets	
$\mathbf{\Sigma}^1_2$		*strict* pwo's on 2^ω / $\mathbf{\Sigma}^1_2$ sets	pwo's on 2^ω / $\mathbf{\Sigma}^1_2$ sets
$\mathbf{\Pi}^1_2$		pwo's on 2^ω	pwo's on $\mathbf{\Pi}^1_2$ sets
			strict pwo's on 2^ω / $\mathbf{\Pi}^1_2$ sets

Future directions

Problem

Does the periodic pattern in the previous diagram continue higher up in the projective hierarchy?

We hope to solve this assuming the axiom of projective determinacy.

We proved partial results on how to characterise those Σ^1_2 sets that admit some countable Σ^1_2-rank.
This leads to a conjecture that generalises results of Kechris, Marker, Sami, Mansfield, Solovay, Stern, Kanovei and Lyubetsky:

Problem

Does every absolutely Δ^1_2 Borel set have a Borel code in L?