Forcing over choiceless models

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What can be done by forcing over arbitrary choiceless models?

This talk is based on joint work with Daisuke Ikegami (Tokyo).
Some results are joint with W. Hugh Woodin (Harvard).

- Daisuke Ikegami, Philipp Schlicht:
  Forcing over choiceless models and generic absoluteness, 28 pages
  to be submitted

Introduction

Mathematics without choice

Set theory without the axiom of choice allows us to do a lot of basic mathematics.

- Many theorems in analysis, for example the intermediate value theorem
- Algebra of countable groups and fields
- Theorems studied in second order arithmetic and reverse mathematics
- Transfinite induction and recursion

Mathematics without choice

However, many things can go wrong:

- Basic measure theory
- Much of functional analysis
- Existence of maximal ideals in rings
- Existence of nontrivial ultrafilters
- Existence of uncountable regular cardinals

Forcing without choice

Choiceless models have been used to separate the axiom of choice from some its consequences such as:

- the ultrafilter lemma (Halpern-Läuchli)
- the existence of a basis for the $\mathbb{Q}$-vector space $\mathbb{R}$.

Steel and Van Wesep introduced forcing over models of determinacy.

Based on this, Woodin developed $\mathbb{P}_{\max}$-forcing over models of determinacy. $L(\mathbb{R})$ and its $\mathbb{P}_{\max}$-extension have canonical theories.

Forcing without choice

Blass proved the following in extensions by a Levy collapse of an inaccessible: An ultrafilter on ω is Ramsey if and only if it is generic for $P(\omega)/\mathrm{fin}$ over $L(\mathbb{R})$.

This was extended in work of Laflamme and Todorcevic.

Forcing without choice

There has been some research on forcing over arbitrary choiceless models.

Monro studied preservation of fragments of the axiom of choice.

Karagila, Schlicht 2020 studied when $\mathrm{Add}(A,1) = \{p \mid p\colon A \rightharpoonup 2 \text{ finite}\}$ adds new reals.

Forcing without choice

What can go wrong?

- Countably closed forcings can collapse ω_1 (folklore).
- Karagila, Schweber 2022: c.c.c. forcings can collapse ω_1.
- Karagila, Schilhan 2022: A forcing may add no new ω-sequences of ordinals, while it is not countably distributive.

  A forcing is called countably distributive if the intersection of countably many open dense sets is dense.
- Boolos 1974 (or folklore): DC holds if and only if every structure has a countable elementary substructure. Thus the definition of proper forcing is not useful if DC fails.

Forcing without choice

The goal is to develop a general theory of forcing over choiceless models. We want to allow failures of even weak choice principles such as DC and AC_ω.

For each forcing or class of forcings, we want to understand what it can do.

- Can one force anything interesting at all over arbitrary choiceless models?

Example I: Cohen's first choiceless model

$\mathrm{Add}(\omega, \omega) = \{p \mid p\colon \omega \times \omega \to 2 \text{ finite}\}$ adds a Cohen subset of A.

Example

$\mathrm{Add}(\omega, \omega)$ adds a sequence $\vec{a} = \langle a_n \mid n \in \omega \rangle$ of Cohen reals.
Let $A = \{a_n \mid n \in \omega\}$ and $V(A)$ the least model $M \supseteq V$ of ZF with $A \in M$.

- DC fails in $V(A)$, since A does not have a countably infinite subset.

Example

Gitik constructed a model of ZF where:

- All uncountable cardinals have countable cofinality.

The construction uses a proper class of strongly compact cardinals.

Remark

If ω_1 is singular, then AC_ω and therefore DC fails:

Proof.

Suppose not. Let $\vec{\alpha} = \langle \alpha_n \mid n \in \omega \rangle$ be cofinal in ω_1.

- Pick $\vec{f}$ with $f_n \colon \alpha_n \to \omega$ injective by AC_ω.

This yields an injective function $f \colon \omega_1 \to \omega$. □

A toolbox

Countably closed forcings

Definition

1. $\mathrm{Col}(\kappa, \lambda) := \{p \colon \alpha \to \lambda \mid \alpha < \kappa\}$.
2. $\mathrm{Col}_*(\kappa, \lambda) := \{(f, g) \mid f \in \mathrm{Col}(\kappa, \lambda),\ g \colon \mathrm{dom}(f) \to |\mathrm{dom}(f)| \text{ bijective}\}$.

$\mathrm{Col}(\kappa, \lambda)$ is ordered by reverse inclusion, while $\mathrm{Col}_*(\kappa, \lambda)$ is ordered by reverse inclusion in the first coordinate.

Remark

If ω_1 is singular, then $\mathrm{Col}(\omega_1, 2)$ is not countably closed.
But $\mathrm{Col}_*(\omega_1, 2)$ is countably closed.

Countably closed forcings

Theorem

TFAE for any set A of size at least 2, $\lambda \in \mathrm{Card}$ and $\mathbb{P} = \mathrm{Col}(\lambda^+, A)$:

1. $\mathsf{DC}_\lambda(A^\lambda)$.
2. $\mathbb{P}$ *is λ-distributive.*
3. $\mathbb{P}$ *does not change V^λ.*
4. $\mathbb{P}$ *preserves size and cofinality of all ordinals $\alpha \leq \lambda^+$.*
5. $\mathbb{P}$ *preserves λ^+ as a cardinal.*
6. $\mathbb{P}$ *forces that λ^+ is regular.*

The same equivalences hold for $\mathrm{Col}_*(\lambda^+, A)$.

Linked forcings

Let $\mathbb{C}^\kappa$ denote the finite support product of κ many Cohen forcings $\mathbb{C} = \{p \mid p\colon n \rightharpoonup 2, n \in \omega\}$.

Karagila observed that wellordered c.c.c. forcings such as $\mathbb{C}^\kappa$ preserve cardinals.

We can reduce finite support products and (uniform) iterations of σ-linked forcings fo $\mathbb{C}^\kappa$ to show they also preserve cardinals.

- A forcing $\mathbb{P}$ is called $\mathbb{Q}$-linked if there is a $\perp$-homomorphism from $\mathbb{P}$ to $\mathbb{Q}$.
- We equip each ordinal θ with the discrete partial order.

Linked forcings

We call a product or iteration of σ-linked forcings uniform if it comes with a sequence of names for linking functions.

Theorem

A uniform finite support product or iteration of σ-linked forcings of length κ is $\mathbb{C}^\kappa$-linked.

Any $\mathbb{C}^\kappa$-linked forcing preserves cardinals.

This a special case of the following notion.

Narrow forcings

Definition

Suppose that $\mathbb{P}$ is a forcing and θ, ν are ordinals, where θ is infinite.

1. $\mathbb{P}$ is called (θ, ν)-narrow if for any ordinal $\mu \leq \nu$ and any sequence $\vec{f} = \langle f_i \mid i < \mu \rangle$ of partial $\|$-homomorphisms $f_i \colon \mathbb{P} \to \mathrm{Ord}$,
$$|\bigcup_{i<\mu} \mathrm{ran}(f_i)| \leq |\max(\theta, \mu)|.$$
2. $\mathbb{P}$ is called θ-narrow if it is (θ, ν)-narrow for all $\nu \in \mathrm{Ord}$. It is called narrow if it is ω-narrow.

We further call $\mathbb{P}$ uniformly (θ, ν)-narrow if there exists a function G_ν that sends each sequence $\vec{f} = \langle f_i \mid i < \mu \rangle$ of partial $\|$-homomorphisms $f_i \colon \mathbb{P} \to \mathrm{Ord}$,[1] where $\mu \leq \nu$, to an injective function
$$G_\nu(\vec{f}) \colon \bigcup_{i<\mu} \mathrm{ran}(f_i) \to \max(|\theta|, \mu).$$
It is called uniformly narrow if it is uniformly ω-narrow.

[1] We can assume $\mathrm{ran}(f_i)$ is an ordinal.

Narrow forcings

Lemma

Every (θ,ν)-narrow forcing $\mathbb{P}$ preserves all cardinals and cofinalities in the interval $(\theta,\nu^+]$.

Lemma

Suppose that θ, ν are cardinals, where θ is infinite, and $f\colon \mathbb{P}\to\mathbb{Q}$ is a $\perp$-homomorphism.

1. *$\mathbb{Q}$ is (θ,ν)-narrow, then $\mathbb{P}$ is (θ,ν)-narrow.*
2. *$\mathbb{Q}$ is uniformly (θ,ν)-narrow, then $\mathbb{P}$ is uniformly (θ,ν)-narrow.*

Narrow forcings

Theorem

Suppose that $\theta \leq \nu$ are infinite ordinals. Any uniform iteration of (θ, ν)-narrow forcings with finite support is again uniformly (θ, ν)-narrow.

This allows us to iterate a mix of Cohen forcing, Hechler forcing and random algebras while preserving all cardinals and cofinalities.

Random algebras

An α-Borel code for a subset of 2^α is a subset of α that codes a set formed from basic open subsets of 2^α via complements and countable unions. Let $2^{(\alpha)} = \{f \mid f\colon \alpha \rightharpoonup 2 \text{ finite.}\}$.

$\mathbb{R}_\alpha$ denotes the forcing that consists of all Borel codes for subsets of 2^α ordered by $\leq$. The quotient of $\mathbb{R}_\alpha$ by $=_\mu$ with the operations induced by $\vee$, $\wedge$ and $-$ is a Boolean algebra.

A forcing is called complete if every subset has a supremum. To show $\mathbb{R}_\alpha$ is complete, we associate to every $A \in \mathbb{R}_\alpha$ its footprint $\mathfrak{f}_A = \langle \mathfrak{f}_{A,t} \mid t \in 2^{(\alpha)} \rangle$, where $\mathfrak{f}_{A,t}$ denotes the relative measure:

$$\mathfrak{f}_{A,t} := \frac{\mu([p] \cap N_t)}{\mu(N_t)}.$$

Let $\mathfrak{f}_A \leq \mathfrak{f}_B$ if $\mathfrak{f}_{A,t} \leq \mathfrak{f}_{B,t}$ for all $t \in 2^{(\alpha)}$. Note that $A \leq B$ if and only if $\mathfrak{f}_A \leq \mathfrak{f}_B$.

Definition

Suppose that $\vec{\mathfrak{f}} = \langle \mathfrak{f}_s \mid s \in 2^{(\alpha)} \rangle$ is a sequence in $\mathbb{R}$ and $x \in 2^\alpha$.

1. For any $\epsilon > 0$, x is called an ϵ-density point of $\mathfrak{f}$ if

$$\exists s \, \forall t \supseteq s \; \mathfrak{f}_t > 1 - \epsilon.$$

2. x is called a density point of $\mathfrak{f}$ if it is an ϵ-density point of $\mathfrak{f}$ for all $\epsilon \in \mathbb{Q}^+$.

The α-Borel code induced by 2 is denoted $D(\mathfrak{f})$.

Random algebras

To construct a least upper bound, we first form the least upper bound of the footprints: let $\mathrm{f}_{X,t} := \sup_{A \in X} \mathrm{f}_{A,t}$ for each $t \in 2^{(\alpha)}$ and

$$\mathrm{f}_X := \langle \mathrm{f}_{X,t} \mid t \in 2^{(\alpha)} \rangle.$$

Lemma

1. *In any outer model W of V such that α is countable in W, $D(\mathrm{f}_X)$ is a least upper bound for X.*
2. *$\mathbb{R}_\alpha$ is complete. More precisely, for any subset X of $\mathbb{R}_\alpha$ the reduct of $D(\mathrm{f}_X)$ is a least upper bound for X.*

The reduct is defined by induction on the rank by reducing each union by a countable one.

Using completeness, we can show random algebras are uniformly narrow.

Theorem

Suppose that κ is a cardinal of uncountable cofinality. Then $\mathbb{H}^{(\kappa)}$ forces $\mathfrak{b} = \mathfrak{d} = \mathrm{cof}(\kappa)$.

Theorem

Suppose $\nu \geq \omega_1$ is multiplicatively closed and has countable cofinality. Any uniform iteration $\mathbb{P}_\nu$ of nontrivial forcings with finite support of length ν forces:

1. $\mathfrak{b} = \omega_1$ *if $\mathbb{P}_\nu$ preserves ω_1.*
2. $\mathfrak{d} \geq |\nu|$ *if $\mathbb{P}_\nu$ preserves $|\nu|$ and $\mathfrak{d}$ exists in the extension.*

In particular, this holds for $\mathbb{H}^{(\nu)}$.

Absoluteness

Let $M \equiv N$ denote that M and N are elementarily equivalent, i.e., they have the same theories.

Definition

The unrestricted absoluteness principle $\mathsf{A}_{\mathcal{C}}$ for a class $\mathcal{C}$ of forcings states that $V \equiv V[G]$ for any generic extension of V by a forcing in $\mathcal{C}$.

Lemma (folklore)

If x is a Cohen real over $L[y]$ where y is a real, then y is not a random real over $L[x]$.

Theorem (Woodin)

Suppose κ is an uncountable cardinal. If H is $\mathbb{C}^\kappa$-generic over V then in $V[H]$, there exists a subset A of ω_1 such that there exists no random real over $L[A]$.

So for any $\kappa \geq \omega_2$, $\mathbb{C}^\kappa$- and $\mathbb{R}_\kappa$-generic extensions have different theories.

Absoluteness

Let $\mathbb{H}^{(*)}$ denote the class of finite support iterations of Hechler forcing.

Theorem

$\mathsf{A}_{\mathbb{H}^{(*)}}$ *implies that all infinite cardinals have countable cofinality.*

Proof.

First suppose that there exists some regular $\kappa \geq \omega_2$. Then $\mathbb{H}^{(\kappa)}$ forces $\mathfrak{b} = \kappa$. Moreover, $\mathbb{H}^{(\aleph_\omega)}$ forces $\mathfrak{b} = \omega_1$ by the above theorem. This contradicts $\mathsf{A}_{\mathbb{H}^{(*)}}$.

Now suppose ω_1 is regular. Then $\mathbb{H}^{(\omega_1)}$ forces $\mathfrak{d} = \omega_1$. However, $\mathbb{H}^{(\aleph_\omega)}$ forces that there exists no dominating family of size ω_1. Again, this contradicts $\mathsf{A}_{\mathbb{H}^{(*)}}$. $\square$

Absoluteness

Write $A \leq_i B$ if there exists an injective function from A into B. Let

$$\mathfrak{c} := \sup\{\lambda \in \mathrm{Card} \mid \lambda \leq_i 2^\omega\}.$$

Remark

We claim that $\mathbb{C}^\nu$ forces $\mathfrak{c} = \nu$ for any ω-strong limit cardinal ν of uncountable cofinality.

To see this, show $\nu^+ \leq_i P_{\omega_1}(\nu)$ using nice names for reals.

Since $\mathrm{cof}(\nu) \geq \omega_1$, we have $\nu \leq_i P_{\omega_1}(\mu)$ for some $\mu < \nu$, contradicting that ν is an ω-strong limit.

Remark

$\mathsf{A}_{\mathbb{C}^*}$ implies that there cannot exist two distinct uncountable regular cardinals $\kappa < \lambda$. Otherwise we would have $\mathrm{cof}(\mathfrak{c}) = \kappa$ and $\mathrm{cof}(\mathfrak{c}) = \lambda$ in some $\mathbb{C}^{(*)}$-generic extensions, contradicting $\mathsf{A}_{\mathbb{C}^*}$.

Theorem (Woodin)

$\mathsf{A}_{\mathbb{C}^*}$ *implies that all infinite cardinals have countable cofinality.*

The main step shows $1_{\mathbb{C}^\kappa} \Vdash \mathfrak{c} > \kappa$ for any ω-strong limit cardinal κ.

If there exist uncountable regular cardinals, then the previous remark yields a contradiction.

Gitik's model

Gitik's model:

- All infinite cardinals have countable cofinality.
- Constructed from a proper class of strongly compact cardinals.
- For each strongly compact κ and $\alpha \geq \kappa$, one can give α countable cofinality using a strongly compact Prikry forcing at κ.
- The symmetric model contains all such Prikry sequences.

Problem:

- λ is a singular limit of strongly compacts $\langle \kappa_\beta \mid \beta < \mathrm{cof}(\lambda) \rangle$ and α is the next inaccessible. One combines all κ_β in the forcing at α to ensure no bounded subsets of λ are added.

$\mathbb{P}_s$ denotes the restriction of Gitik's forcing $\mathbb{P}$ to a finite set $s \subseteq \mathrm{Ord}$.

Lemma (Gitik)

For sufficiently closed finite $s \subseteq \mathrm{Ord}$ and strongly compact $\kappa_\xi \in s$, $\mathbb{P}_s$ is forcing equivalent to a forcing of the form $\mathbb{P}_{s \cap \kappa_\xi} * \dot{\mathbb{Q}}$, where:

- $\mathbb{P}_{s \cap \kappa_\xi}$ *has size* $\leq \kappa_\xi$.
- $\mathbb{P}_{s \cap \kappa_\xi}$ *forces that* $\dot{\mathbb{Q}}$ *adds no bounded subsets of* κ_ξ.

Gitik's model

Let

$$\mathsf{c}_\kappa = \sup\{\lambda \in \mathrm{Card} \mid \lambda \leq_i \kappa^\omega\}.$$

- In Gitik's model, $\mathsf{c}_\kappa = \kappa$ holds for all infinite cardinals κ using the previous lemma
- $\mathsf{A}_{\mathbb{C}*}$ implies $\mathsf{c}_\kappa = \mathsf{c}_\omega^{V^{\mathbb{C}\kappa}} > \kappa$ for all ω-strong limit cardinals.

Hence $\mathsf{A}_{\mathbb{C}*}$ fails in Gitik's model

Open questions

Is $A_{\mathbb{C}^*}$ consistent?

Can we produce more switches, for instance by separating Hechler from Cohen models?

Regarding the notion of narrow forcing, do $(\omega, 1)$-narrow forcings preserve cardinals?

Gitik's model is an interesting test case. Do the classical tree forcings preserve ω_1 over this model?