Interaction of determinacy and forcing

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Two themes

Projective determinacy is a useful axiom for studying definable sets of reals beyond Borel and analytic sets.

Forcing is an important technique to study the independence of properties of sets of reals.

Problem

Which forcings preserve projective determinacy?

This is based on joint work with Jonathan Schilhan (Leeds) and Johannes Schürz (Vienna).

Test question

Question

Does Cohen forcing preserve analytic determinacy?

Determinacy

Fix a subset A of 2^ω. In the game $G(A)$, two players I and II alternate playing moves with values 0 and 1.

$$\begin{array}{c|cccccccccc} \text{I} & i_0 & & i_2 & & i_4 & & i_6 & & \dots \\ \hline \text{II} & & i_1 & & i_3 & & i_5 & & i_7 & \dots \end{array}$$

$$\text{II wins the run} \iff x = \langle i_n \mid n \in \omega \rangle \in A$$

$G(A)$ is called determined if I or II has a winning strategy.

Projective determinacy

Theorem (Martin 1975)

All Borel sets are determined.

Projective determinacy (PD) is the statement that all projective sets are determined.

- An analytic or $\mathbf{\Sigma}^1_1$ set is the projection $p[C]$ of a closed subset C of $\omega^\omega \times \omega^\omega$ to the first coordinate.
- $\mathbf{\Pi}^1_1$ sets are complements of $\mathbf{\Sigma}^1_1$ sets.
- $\mathbf{\Sigma}^1_2$ sets are projections of $\mathbf{\Pi}^1_1$ sets etc.
- A set is projective if it is $\mathbf{\Sigma}^1_n$ for some n.

Theorem (Mycielski, Swierczkowski 1964, Moschovakis 1971)

Assume projective determinacy.

1. *All projective sets are Lebesgue measurable.*
2. *Every projective binary relation R on 2^ω has a projective uniformisation.*
   A uniformisation of R is a subfunction of R with domain $p[R]$.

What was known

It is easy to destroy determinacy if ω_1 can be collapsed. We will thus assume $\mathbb{P}$ is proper.

In fact, we will always assume stronger forms of properness. It is open whether properness suffices.

It is further natural to assume $\mathbb{P}$ is a projective forcing on the reals. For analytic determinacy, the complexity has to be low.

Theorem (David 1978)

It is consistent that there exists a Σ^1_3-definable proper forcing that destroys analytic determinacy.

What was known

Analytic determinacy is closely linked with Σ^1_3-absoluteness.

Theorem (Woodin 1982)

Analytic determinacy (actually uniformisation up to meager resp. null) implies Σ^1_3-absoluteness for Cohen and random forcing.

The proofs of preservation of projective determinacy also show projective absoluteness.

(Does preservation of projective determinacy imply projective absoluteness?)

What was known

Problem

Does every Borel proper forcing preserve analytic determinacy?

This is really a question of Ikegami (2010). He asked this for absolutely Δ^1_2 proper tree forcings.

Theorem (S. 2014)

Any Σ^1_2 absolutely c.c.c. forcing preserves Π^1_n-determinacy for each $n \geq 1$.

Theorem (Castiblanco, S. 2020)

Sacks, Mathias, Laver and Silver forcing preserve Π^1_n-determinacy for each $n \geq 1$.

Related results

Theorem (essentially Judah, Shelah 1989, Pawlikowski 1986)

1. *Cohen forcing preserves the statement: all Δ^1_2 sets have the Baire property.*
2. *Random forcing preserves the statement: all Δ^1_2 sets are Lebesgue measurable.*

Theorem (Judah, Shelah 1988, Goldstern, Shelah 1992)

Over Solovay models, countable support iterations of Suslin proper-for-candidates forcings preserve the statement: all projective sets have the property of Baire.

A forcing $(\mathbb{P}, \leq)$ on the reals is called Suslin if $\leq$ is analytic.

Proper-for-candidates is a condition for Suslin forcings that implies proper. Elementary submodels $M \prec H_\theta$ are replaced by countable transitive models of a fragment of ZFC.

Background

A cardinal κ is measurable if the following equivalent conditions hold:

- There is a non-principal $<\kappa$-complete ultrafilter on κ.
- There is an elementary embedding $j : V \to N$ to some transitive model N with $\text{crit}(j) = \kappa$.

Theorem (Levy, Solovay 1967)
If κ is measurable and $\mathbb{P}$ is a forcing of size $|\mathbb{P}| < \kappa$, then κ remains measurable in any $\mathbb{P}$-generic extension $V[G]$ of V.

Proof sketch.
Lift $j : V \to N$ to $j^* : V[G] \to N[G]$ by letting $j^*(\sigma^G) = j(\sigma)^G$. $\square$

Many variants of this theorem are known, for example for strong, Woodin, supercompact cardinals.

Background

Some large cardinal properties of small cardinals are preserved by sufficiently nice forcings.

Theorem (Foreman 2013)

Generic supercompactness of ω_1 is preserved by all proper forcings.

Definition (Silver et al.)

$0^{\#}$ exists if (equivalently) each of the following objects exist:

1. An uncountable set of ordinals which are order-indiscernible over L.
2. A non-trivial elementary embedding $j : L \to L$.
3. A countable structure $(L_\alpha, \in, U)$ such that
   - $(L_\alpha, \in)$ is a model of ZFC$^-$ with a largest cardinal κ,
   - $(L_\alpha, \in, U) \models \Sigma_0$-separation + U is a $<\kappa$-complete ultrafilter on κ
   - All iterated ultrapowers of $(L_\alpha, \in, U)$ are wellfounded.

   The least such structure is denoted $M_0^{\#}$ or $0^{\#}$.

More generally, $x^{\#}$ is defined for any real x by replacing L with $L[x]$.

A measurable cardinal implies that $x^{\#}$ exists for all reals x.

Analytic determinacy and sharps

Theorem (Martin 1970, Harrington 1978)

The following conditions are equivalent:

1. $x^{\#}$ *exists for all reals* x
2. *Analytic determinacy*

Cohen forcing

Proposition (folklore?)

Cohen forcing preserves analytic determinacy.

Proof.

Suppose:

- $x^\#$ exists for all reals x.
- $V[G]$ is a Cohen extension of V. Let x denote the Cohen real.
- σ is a name for a new real. We can assume σ is a nice name.

The name σ is essentially a real, since Cohen forcing has the c.c.c. Thus $\sigma^\#$ exists. Hence there is a nontrivial elementary embedding $j\colon L[\sigma] \to L[\sigma]$.

> *x is Cohen generic over $L[\sigma]$, since Cohen forcing has the c.c.c. and is Σ^1_2-definable.*

We can lift j to $j^* : L[\sigma][G] \to L[\sigma][G]$ as in the Levy-Solovay theorem. Since the new real $\sigma^G \in L[\sigma][G]$, this yields $(\sigma^G)^\#$ in $V[x]$. □

Proposition (Castiblanco, S. 2020)

Sacks forcing $\mathbb{P}$ preserves analytic determinacy.

Proof sketch.

Again, we obtain a small $\mathbb{P}$-name σ and a nontrivial elementary embedding $j \colon L[\sigma] \to L[\sigma]$.

> *Force over $L[\sigma]$ in V with finite subtrees of $2^{<\omega}$ ordered by end extension.*

This adds a perfect tree T such that all its branches are Cohen reals over $L[\sigma]$. This remains true in generic extensions of V.

Force with $\mathbb{P}$ below $T \in \mathbb{P}$ over V. Let x denote the Sacks real.

> *Then x is Cohen generic over $L[\sigma]$.*

Again, we can lift j to $j^* \colon L[\sigma][x] \to L[\sigma][x]$ and obtain $(\sigma^x)^\#$ in $V[x]$. □

Definition (Shelah 1980)

1. Suppose that $\mathbb{P}$ is a forcing, $\dot{G}$ is a $\mathbb{P}$-name for the $\mathbb{P}$-generic filter and M is a model. A condition p is called $(M, \mathbb{P})$-generic if

$$q \Vdash \text{“}\dot{G} \cap N \text{ is } \mathbb{P} \cap N \text{ generic over } N\text{”}.$$

2. $\mathbb{P}$ is proper if for sufficiently large regular θ, there exists a club of countable $M \prec H_\theta$ with the following property.

“For every $p \in \mathbb{P} \cap M$, there exists some $(M, \mathbb{P})$-generic $q \leq p$.”

Suslin forcing

Definition (Goldstern 1992)

1. A Suslin forcing is a forcing on the reals with a Σ^1_1 definition of $\leq$.
2. A strongly Suslin forcing is a Suslin forcing with a Σ^1_1 definition of $\perp$.

Definition (Judah, Shelah 1988)

Let $\mathbb{P}$ be a Suslin forcing.

1. A countable transitive model N is called a *candidate for* $\mathbb{P}$ if $\mathbb{P} \cap N \in N$.
2. $\mathbb{P}$ is called proper-for-candidates[1] if for every candidate N for $\mathbb{P}$ and every $p \in \mathbb{P} \cap N$ there exists an $(N, \mathbb{P})$-generic condition $q \leq p$.

[1] Originally called proper Suslin forcing

Capturing

Definition

1. An operator is a function $\mathbb{M}$ that sends each real x to a structure $\mathbb{M}(x) = (M(x), \in, E)$ such that
    - $x \in M(x)$, $M(x)$ is transitive, $M(x) \models \mathrm{ZFC}^-$ and $M(x)$ is E-amenable.
2. $\mathbb{M}$ is called $\mathbb{Q}$-amenable if each $\mathbb{M}(x)$ is $\mathbb{Q}$-*amenable*.

For example, $\mathbb{M}(x) = (L(x), \in)$.

Definition (Castiblanco, S. 2020)

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcings and M is a $\mathbb{Q}$-amenable operator. We say that $\mathbb{P}$ is captured by $\mathbb{Q}$ over M if the following holds for any $\mathbb{P}$-name τ for a real

" *If H is $\mathbb{P}$-generic over V, then τ^H is contained in a $\mathbb{Q} \cap M(y)$ generic extension of some $M(y)$.* "

Equivalently: For any $p \in \mathbb{P}$ and any real x, there exists a real y with $x \in L(y)$ such that some $q \leq_{\mathbb{P}} p$ forces:

"*There exists a $\mathbb{Q} \cap M(y)$ generic filter g over $M(y)$ with $\tau \in M(y)[g]$.*"

Capturing

All proper-for-candidates Suslin forcings are captured over L. This includes most classical proper forcings which add a real.

Capturing over L implies preservation of analytic determinacy, for proper forcings on the reals.

Iterable structures

One can reformulate the above preservation proofs using iterable structures $(M, \in, U)$ instead of elementary embeddings $j \colon L(\sigma) \to L(\sigma)$.

The Cohen or Sacks real x over V is generic over some $(M_0, \in, U)$. The iteration lifts step by step to

$$M_0[x] \to M_1[x] \to \cdots \to M_\alpha[x] \to \dots$$

Iterable structures

More generally, work with transitive structures $(M, \in, E)$ where

- $(M, \in) \models \mathrm{ZFC}^-$
- E is an M-amenable sequence of (partial) extenders
- All extenders in E, except possibly the last one, are elements of M

A extender is a directed system of ultrafilters.

Definition

We call $(M, \in, E)$ ω_1-iterable if all ultrapowers in countable iteration trees (Martin, Steel) on M using E, for some strategy choosing branches, are wellfounded.

Definition

We call $(M, \in, E)$ n-tall if M has n Woodin cardinals and a measurable cardinal above them, witnessed by E. [2]

An operator $\mathbb{M}$ is called n-tall if each $\mathbb{M}(x)$ is n-tall.

[2] In $(M, \in, U)$, $\mathrm{crit}(U)$ is considered measurable.

Characterisations of PD

The next theorem arises from results of Harrington, Martin, Steel, Woodin and Neeman.

Theorem (essentially HMNSW)
The following statements are equivalent for any n:

1. Π^1_{n+1}*-determinacy.*
2. *There exists an n-tall stably* $\mathbb{M}$*-iterable operator* $\mathbb{M}$.
3. *There exists an n-tall* ω_1*-iterable operator* $\mathbb{M}$.

- We reformulated this using the notion of $\mathbb{N}$-iterability. This means (essentially): $\mathbb{N}(x)$ can compute branches of it. trees $T \in \mathbb{N}(x)$.
- For example, the $\mathbb{M}_n^\#$-operator from inner model theory is stably $\mathbb{M}_k^\#$-iterable for any $k \geq n-1$.

Preservation of PD

Proposition (Schilhan, Schürz, S. 2021)
Suppose that $\mathbb{M}$ is any stably $\mathbb{M}$-iterable $\mathbb{P}$-amenable operator and $\mathbb{P}$ is captured over $\mathbb{M}$.

1. In any $\mathbb{P}$-generic extension, $\mathbb{M}$ can be extended to an $\mathbb{M}^*$-iterable operator $\mathbb{M}^*$.
2. If $\mathbb{M}$ is n-tall, then $\mathbb{M}^*$ is n-tall.

It follows that any proper-for-candidates Suslin forcing preserves PD level by level.

Lemma (Schilhan, Schürz, S. 2021)

Let $\vec{\mathbb{P}}$ is a Suslin forcing such that for every candidate M:

1. $\vec{\mathbb{P}}^M = \vec{\mathbb{P}} \cap M$.
2. $M \models$ "$\vec{\mathbb{P}} \cap M$ *satisfies Axiom A*".
3. *For countable sets* $A \subseteq \mathbb{P}$ *and* $p \in \mathbb{P}$, *the statements*

   "A is an antichain"

   "A is predense below p"

   are absolute between M and V.

Then $\mathbb{P}$ *is proper for candidates.*

Theorem (Schilhan, Schürz, S. 2021)

Let $\mathbb{M}$ be an operator such that ω_1 is inaccessible in each $\mathbb{M}(x)$. Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha < \kappa \rangle$ is a countable support iteration of proper-for-candidate Suslin forcing notions $\mathbb{P}_\alpha$ such that for every $\alpha < \kappa$,

> $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{P}}_\alpha$ *is proper-for-candidates in every small generic extension of any $\mathbb{M}(x)$.*

Then $\mathbb{P}$ is captured over $\mathbb{M}$ by a countable support iteration of Suslin proper-for-candidates forcings of countable length.

Corollary (Schilhan, Schürz, S. 2021)

If analytic determinacy is consistent, then so is its combination with the Borel conjecture.

Theorem (Schilhan, Schürz, S. 2021)
If $\mathbb{P}$ satisfies a uniform version of capturing for Cohen forcing, then $\mathbb{P}$ preserves the statement:

Every Δ^1_2 set has the Baire property.

For example: any countable support product or iteration of Sacks forcing works.

Results

The above results immediately imply projective absoluteness.

The next result talks about thin graphs.

We have general results for thin equivalence relations, but their proofs use transitivity.

Theorem (Schilhan, Schürz, S. 2021)
Suppose that analytic determinacy holds and G is an absolutely Δ^1_3 *thin graph. After forcing with a countable support iteration of Sacks forcing, any new real has an edge to some ground model real.*

Future directions

It is natural to aim for stronger determinacy principles using other operators.

It is open (to my knowledge) whether proper implies proper-for-candidates for Borel forcings (possibly assuming analytic determinacy).