Open dihypergraphs on generalized Baire spaces

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- Philipp Schlicht, Dorottya Sziráki:
  The open dihypergraph dichotomy for generalized Baire spaces,
  70 pages, in preparation

Motivation: Ramsey theory

It is natural to wonder whether Ramsey's theorem for n-tuples of natural numbers can be extended to the set of real numbers.

- Sierpinski's counterexample 1937: a partition of pairs of reals in two pieces with no uncountable homogeneous set
- Galvin 1968: Ramsey's theorem for open graphs on the reals
- Blass 1981: A generalization to Borel n-hypergraphs on the reals

Galvin's theorem can be strengthened.

- Todorcevic's open graph axiom 1989:
- Feng's open graph dichotomy for analytic sets 1993:

Feng's theorem implies one of the most basic descriptive set theoretic dichotomies: the perfect set property.

The graph-theoretic approach

In the last few years, graph dichotomies provided new proofs of old and new theorems in descriptive set theory.

Kechris, Solecki, Todorcevic and Miller proved results for analytic graphs (variants of the G_0-dichotomy) that imply:

- Suslin's perfect set property of analytic sets
- Lusin and Novikov's uniformization of Borel sets with countable sections
- Feng's open graph dichtomy
- Silver's theorem on coanalytic equivalence relations

The graph-theoretic approach

Carroy, Miller and Soukup 2020 found an infinite dimensional version of Feng's open graph dichotomy.

Note the following restrictions:

- Farah, Todorcevic 1995: The open graph dichotomy fails for closed graphs.
- Farah, Todorcevic 1995, He 2005: The open 3-hypergraph dichotomy fails.

One thus has to consider directed hypergraphs.

A κ-dihypergraph on X is a set of nonconstant sequences in ${}^{\kappa}X$.

The open graph dichotomy

A graph G is a symmetric relation with no loops.

A graph G on a space X is an open graph if it is an open subset of $X \times X$ without the diagonal.

Definition (Feng 1993)

$\mathrm{OGD}_\omega(X)$ states that for any open graph G on X, either

1. G has an ω-coloring or
2. G has a perfect complete subgraph.

G has an ω-coloring if and only if X is the union of countably many G-independent sets.

We fix the box topology on ${}^{\omega}X$ with basic open sets $\prod_{i<\omega} U_i$, where each U_i is open in X.

Definition (Carroy, Miller, Soukup 2020)

$\mathrm{ODD}^{\omega}_{\omega}(X)$ states that for any box-open ω-dihypergraph H on X, either

1. H has a ω-coloring or
2. there is a continuous homomorphism $f : {}^{\omega}\omega \to X$ from $\mathbb{H}_{\omega_\omega}$ to H.

$$\mathbb{H}_{\omega_\omega} = \left\{\vec{x} \in {}^{\omega}({}^{\omega}\omega) \mid \exists t \in {}^{<\omega}\omega \ \ \forall n \in \omega \ t^\frown\langle n\rangle \subseteq x_n\right\}$$

$\mathrm{ODD}^{\omega}_{\omega}(X, H)$ states that this holds for H.

Applications

Theorem (CMS)
$\mathrm{ODD}^{\omega}_{\omega}(X)$ *holds for all analytic subsets* X *of* ${}^{\omega}\omega$.
It holds for all subsets, assuming AD.

They prove a number of applications:

1. The Hurewicz dichotomy for X: either
   - X is contained in a K_σ set, or
   - X contains a closed subset homeomorphic to ${}^{\omega}\omega$.
2. The Jayne-Rogers theorem on piecewise continuous functions with closed pieces on X.
3. A theorem of Lecomte and Zeleny on $\mathbf{\Delta}^0_2$-measurable ω-colorings on X.

For a metric space X, let H_X denote the ω-dihypergraph on X of all injective sequences in X with no convergent subsequence.

Proposition (CMS)

1. H_X is box-open.
2. There is an ω-coloring of $H_X \restriction Y$ iff Y is contained in a K_σ set.
3. A continuous function ${}^\omega\omega \to X$ is a homomorphism from $\mathbb{H}_{\omega_\omega}$ to H_X iff it is an injective closed map.

Proof sketch.
For 2., note that a subset Y of X is H_X-independent iff its closure is compact. $\square$

Generalized Baire spaces

κ always denotes an uncountable cardinal with $\kappa^{<\kappa} = \kappa$.
Definitions are analogous:

- The κ-Baire space ${}^\kappa\kappa$ is the set of functions $x : \kappa \to \kappa$ with the bounded topology. The basic open sets are

$$N_t = \{x \in {}^\kappa\kappa \mid t \subseteq x\}$$

  for all $t \in {}^{<\kappa}\kappa$.
- The κ-Cantor space ${}^\kappa 2$ has subspace topology.
- κ-Borel sets are generated from open sets by closing under unions and intersections of size κ and negations.
- κ-analytic sets are continuous images of closed sets.

From PSP to OGD

Relative to an inaccessible cardinal:

Theorem (Lücke, Motto Ros, S. 2016)

The Hurewicz dichotomy for all κ-analytic subsets of ${}^{\kappa}\kappa$ is consistent.

Theorem (S. 2017)

The perfect set property (PSP) for all definable subsets of ${}^{\kappa}\kappa$ is consistent.

By definable we mean definable from a sequence in ${}^{\kappa}\mathrm{Ord}$.

Theorem (Sziraki 2018)

The open graph dichotomy (OGD) for all κ-analytic subsets of ${}^\kappa\kappa$ is consistent.

Definition

$\mathrm{ODD}^\kappa_\kappa(X)$ states that for any box-open κ-dihypergraph H on X, either

1. H has a κ-coloring or
2. there is a continuous homomorphism $f : {}^\kappa\kappa \to X$ from $\mathbb{H}_{{}^\kappa\kappa}$ to H.

$$\mathbb{H}_{{}^\kappa\kappa} = \left\{ \vec{x} \in {}^\kappa({}^\kappa\kappa) \mid \exists t \in {}^{<\kappa}\kappa \;\; \forall i \in \kappa \; t^\frown\langle i\rangle \subseteq x_i \right\}$$

$\mathrm{ODD}^\kappa_\kappa(X, H)$ states that this holds for H.

$\mathrm{ODD}^\alpha_\kappa$ denotes the version for α-dihypergraphs.

Theorem (Sziraki, S. 2021)
Suppose that V is a $\mathrm{Col}(\kappa, {<}\lambda)$*-generic extension. Then* $\mathrm{ODD}^{\omega}_{\omega}(X, H)$ *holds for all definable subsets X of* ${}^{\kappa}\kappa$ *and:*

1. *all definable box-open* κ*-dihypergraphs H on X, if* λ *is inaccessible in the ground model.*
2. *arbitrary box-open* κ*-dihypergraphs H on X, if* λ *is Mahlo in the ground model.*

Applications

- All applications of CMS in the countable case are consistent relative to an inaccessible or Mahlo cardinal. They do not need AD.
- The Hurewicz dichotomy: X contains a closed homeomorphic copy of ${}^{\kappa}\kappa$ or X is contained in a union of κ many κ-compact sets.

Example

$\mathrm{ODD}^2_\kappa(X)$ implies the open graph dichotomy $\mathrm{OGD}_\kappa(X)$.

To see this, take $x \neq y$ in ${}^\kappa 2$. Let $i < \kappa$ be least with $x(i) \neq y(i)$.

$$\langle x, y\rangle \in \mathbb{H}_{{}^\kappa 2} \iff x(i) = 0 \wedge y(i) = 1.$$

The complete graph $\mathbb{K}_{{}^\kappa 2}$ on ${}^\kappa 2$ is the smallest (symmetric) graph containing $\mathbb{H}_{{}^\kappa 2}$.

Thus a continuous homomorphism $f : {}^\kappa 2 \to X$ from $\mathbb{H}_{{}^\kappa 2}$ to a graph G is also a homomorphism from $\mathbb{K}_{{}^\kappa 2}$ to G.

Note that f is injective. So G has a perfect complete subgraph.

Step 1: Reflection

Notation: Let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic, where $\lambda > \kappa$ is inaccessible.

For each $\alpha < \lambda$, let $G_\alpha = G \cap \mathrm{Col}(\kappa, {<}\alpha)$. Write

$$X_{\varphi,a} = \{x \in {}^\kappa\kappa : \varphi(x, a)\}$$

Lemma

Suppose $X \subseteq {}^\kappa\kappa$. If X is definable in $V[G]$ or λ is Mahlo in V, then

$$X \cap V[G_\nu] \in V[G_\nu]$$

for stationarily many $\nu < \lambda$.

Proof sketch.

If X is definable in $V[G]$, the claim holds for a tail of $\nu < \kappa$, since the tail forcings are homogeneous.

Now suppose that λ is Mahlo in V.

Let $\dot{X}$ be a name for X. Define $f : \lambda \to \lambda$ as follows.

For $\alpha < \lambda$ and a nice $\mathrm{Col}(\kappa, <\alpha)$-name $\dot{x} \in V$ for a subset of $\kappa \times \kappa$, let $A_{\dot{x}}$ be a maximal antichain in $\mathrm{Col}(\kappa, <\lambda)$ deciding $\dot{x} \in \dot{X}$.

Since $\mathrm{Col}(\kappa, <\lambda)$ has the λ-c.c., let $f(\alpha) < \lambda$ be such that $A_{\dot{x}} \subseteq \mathrm{Col}(\kappa, <f(\alpha))$ for all such nice names $\dot{x}$.

The set S of inaccessible closure points of f is stationary, since λ is Mahlo.

Claim

$X \cap V[G_\nu] \in V[G_\nu]$ for all $\nu \in S$.

Let

$$F_\nu(\dot{x}^{G_\nu}) = \begin{cases} 1 & \text{if } p \Vdash^V_{\mathrm{Col}(\kappa,<\lambda)} \dot{x} \in \dot{X} \text{ for some } p \in G_\nu, \\ 0 & \text{if } p \Vdash^V_{\mathrm{Col}(\kappa,<\lambda)} \dot{x} \notin \dot{X} \text{ for some } p \in G_\nu. \end{cases}$$

F_ν is the characteristic function of $X \cap V[G_\nu]$, since $G_\nu \subseteq G$.

Step 2: Independent trees

In $V[G]$, suppose $a \in {}^{\kappa}\mathrm{Ord}$. Write

$$X_{\varphi,a} = \{x \in {}^{\kappa}\kappa \mid \varphi(x, a)\}.$$

$$\mathcal{T}^{\mathrm{ind}} = \{T \subseteq {}^{<\kappa}\kappa \mid T \text{ is a tree, } [T] \text{ is } R\text{-independent}\}.$$

Then $\mathcal{T}^{\mathrm{ind}} \cap V[G_\nu] \in V[G_\nu]$ for some $\nu < \lambda$ with $a \in V[G_\nu]$ by the previous step. We can assume $V[G_\nu] = V$.

If R has no κ-coloring, then for some $\gamma < \lambda$:

$$(X_{\varphi,a} \setminus \bigcup\{[T] \mid T \in \mathcal{T}_V^{\mathrm{ind}}\}) \cap V[G_\gamma] \neq \emptyset.$$

Step 2: Independent trees

In V, let $\dot{x}$ be a $\mathrm{Col}(\kappa,{<}\gamma)$-name for an element of $X_{\varphi,a}$ such that $1_{\mathrm{Col}(\kappa,<\gamma)} \Vdash \dot{x} \notin [T]$ for all $T \in \mathcal{T}_V^{\mathrm{ind}}$. For any $p \in \mathrm{Col}(\kappa,{<}\gamma)$, let

$$T^{\dot{x},p} = \{t \in {}^{<\kappa}\kappa \mid \exists q \leq p \;\; q \Vdash t \subseteq \dot{x}\}$$

denote the tree of possible values for $\dot{x}$ below p.

Lemma

1. $1_{\mathrm{Col}(\kappa,<\gamma)} \Vdash$ *“$\dot{x} \in X_{\varphi,a}$ in every further $\mathrm{Col}(\kappa,{<}\lambda)$-gen. extension.”*
2. $T^{\dot{x},p} \notin \mathcal{T}_V^{\mathrm{ind}}$ *for all* $p \in \mathrm{Col}(\kappa,{<}\gamma)$.

Proof of 2. $\quad p \Vdash \dot{x} \in [T^{\dot{x},p}]$. □

We now assume $\dot{x}$ is an $\mathrm{Add}(\kappa,1)$-name.

Step 3: Construction of a forcing

The forcing will construct the required homomorphism. The point is to avoid subsets of κ with bad quotients.

We construct a forcing $\mathbb{Q}$ such that:

1. $\mathbb{Q}$ is equivalent to $\mathrm{Add}(\kappa, 1)$.
2. Suppose that $V[H]$ is any $\mathbb{Q}$-generic extension of V. $\mathbb{Q}$ adds a map $g : ({}^\kappa\kappa)^{V[H]} \to ({}^\kappa\kappa)^{V[H]}$ such that for each $y \in ({}^\kappa\kappa)^{V[H]}$,
    - $g(y)$ is $\mathrm{Add}(\kappa, 1)$-generic over V,
    - $V[H]$ is a $\mathrm{Add}(\kappa, 1)$-generic extension of $V[g(y)]$, and
    - $\dot{x}^{g(y)} \in X_{\varphi,a}$.

   $f : {}^\kappa\kappa \to X$, $f(y) = \dot{x}^{g(y)}$ is continuous.
3. f is a homomorphism from $\mathbb{H}_{{}^\kappa\kappa}$ to R.

The main work is to prove properties of $\mathbb{Q}$.

Future directions

Inaccessibles are necessary for our results.
Are Mahlo cardinals necessary? (They are for the proofs.)
This would separate the variant for arbitrary dihypergraphs from the definable variant.

Do other large cardinals play a role for the structure of definable subsets of generalized Baire spaces?

Regarding more complex graphs, Is a version of the G_0-dichotomy for ${}^{\kappa}\kappa$ consistent?
Is the Lusin-Novikov theorem for ${}^{\kappa}\kappa$ consistent?