Interaction of determinacy and forcing

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Two themes

Projective determinacy is a useful axiom for studying definable sets of reals beyond Borel and analytic sets.

Forcing is an important technique to study the independence of properties of sets of reals.

Problem

Which forcings preserve projective determinacy?

Based on a joint project with Jonathan Schilhan (Leeds) and Johannes Schürz (Vienna)

Test question

Question

Does Cohen forcing preserve analytic determinacy?

Determinacy

Fix a subset A of 2^ω. In the game $G(A)$, two players I and II alternate playing moves with values 0 and 1.

$$\begin{array}{c|cccccccccc} \text{I} & i_0 & & i_2 & & i_4 & & i_6 & & \dots \\ \hline \text{II} & & i_1 & & i_3 & & i_5 & & i_7 & \dots \end{array}$$

$$\text{II wins the run} \iff x = \langle i_n \mid n \in \omega \rangle \in A$$

$G(A)$ is called determined if I or II has a winning strategy.

Projective determinacy

Theorem (Martin 1975)

All Borel sets are determined.

Projective determinacy (PD) is the statement that all projective sets are determined.

- An analytic or $\mathbf{\Sigma}^1_1$ set is the projection $p[C]$ of a closed subset C of $\omega^\omega \times \omega^\omega$ to the first coordinate.
- $\mathbf{\Pi}^1_1$ sets are complements of $\mathbf{\Sigma}^1_1$ sets.
- $\mathbf{\Sigma}^1_2$ sets are projections of $\mathbf{\Pi}^1_1$ sets etc.
- A set is projective if it is $\mathbf{\Sigma}^1_n$ for some n.

Theorem (Mycielski, Swierczkowski 1964, Moschovakis 1971)

Assume projective determinacy.

1. *All projective sets are Lebesgue measurable.*
2. *Every projective binary relation R on 2^ω has a projective uniformisation.*
   A uniformisation of R is a subfunction of R with domain $p[R]$.

Forcing extensions

Suppose that M is a transitive model of ZFC.

Definition

Suppose that $\mathbb{P} \in M$ is a forcing, i.e. a poset with a largest element 1, and G is a subset of $\mathbb{P}$.

1. A $\mathbb{P}$-name $\sigma \in M$ is a set of pairs (τ, p), where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.
2. The evaluation of σ by G is

$$\sigma^G = \{\tau^G \mid \exists(\tau, p) \in \sigma,\ p \in G\}.$$

3. The extension of M by G is

$$M[G] = \{\sigma^G \mid \sigma \in M\}.$$

Cohen forcing

Forcing is easiest to illustrate if $\mathbb{P}$ is a tree.

Example

Cohen forcing is the poset $\mathbb{P} = 2^{<\omega}$ of all finite sequences $p\colon n \to 2$, ordered by reverse inclusion: $p \leq q$ if p extends q.

Then G is chosen as a branch. The function $x_G = \bigcup_{p \in G} p$ is called the Cohen real.

For posets, the analogue to a branch is a filter. G is always assumed to be a generic filter.

Sacks forcing

$M[G]$ equals the least transitive model of ZFC that contains M as a subclass and G as an element.

In fact $M[G] = M[x_G]$.

There are many other classical forcings that add a real and have this property.

Example

Sacks forcing is the poset $\mathbb{P}$ consisting of all perfect subtrees of $2^{<\omega}$ ordered by inclusion: $p \leq q$ if p is a subtree of q.

Cohen and Sacks forcing can also be unterstood as a set of reals.

The story of proper forcing

We want to preserve ω_1.

The problem is that iterating ω_1-preserving forcings, i.e. forming iterated forcing extensions, may collapse ω_1.

The countable chain condition (c.c.c.) states that $\mathbb{P}$ does not have uncountable antichains. This suffices to preserve ω_1 in iterations.

Cohen forcing has the c.c.c., but Sacks forcing does not.

It was known at least since Baumgartner that Sacks forcing can be iterated without collapsing ω_1.

He proved this for Axiom A forcings. This class extends both c.c.c. and σ-closed.

Shelah isolated the more general notion of proper forcing.

$\mathbb{P}$ is called proper if it preserves stationary subsets of $[\lambda]^\omega$ for all uncountable λ.

What was known

Since it is easy to destroy determinacy if ω_1 can be collapsed, we assume $\mathbb{P}$ is proper.

Theorem (David 1978)

It is consistent that some Σ^1_3-definable proper forcing destroys analytic determinacy.

We therefore need to assume $\mathbb{P}$ is simply definable.

Analytic determinacy is closely linked with Σ^1_3-absoluteness.

Theorem (Woodin 1982)

Analytic determinacy (actually uniformisation up to meager resp. null) implies Σ^1_3-absoluteness for Cohen and random forcing.

What was known

Problem (Ikegami 2010)

Does every Borel proper forcing preserve analytic determinacy?[1]

Theorem (S. 2014)

Any Σ^1_2 *absolutely c.c.c. forcing preserves* Π^1_n*-determinacy for each* $n \geq 1$.

Theorem (Castiblanco, S. 2020)

Several classical tree forcings preserve Π^1_n*-determinacy for each* $n \geq 1$.

[1] He asked this for absolutely Δ^1_2 proper forcings.

A related result

Theorem (Judah, Shelah 1988, Goldstern, Shelah 1992)

Over Solovay models, countable support iterations of Suslin proper-for-candidates forcings preserve the property of Baire for all projective sets.

A forcing $(\mathbb{P}, \leq)$ on the reals is called Suslin if $\leq$ is analytic.

Proper-for-candidates is a condition that implies proper. Elementary submodels $M \prec H_\theta$ are replaced by countable transitive models of a fragment of ZFC.

A cardinal κ is measurable if the following equivalent conditions hold:

- There is a non-principal $<\kappa$-complete ultrafilter on κ.
- There is an elementary embedding $j : V \to N$ to some transitive model N with $\mathrm{crit}(j) = \kappa$.

Theorem (Levy, Solovay 1967)
If κ is measurable and $\mathbb{P}$ is a forcing of size $|\mathbb{P}| < \kappa$, then κ remains measurable in any $\mathbb{P}$-generic extension $V[G]$ of V.

Proof sketch.
Lift $j : V \to N$ to $j^* : V[G] \to N[G]$ by letting $j^*(\sigma^G) = j(\sigma)^G$. □

Many variants of this theorem are known, for example for strong, Woodin, supercompact cardinals.

Background

Some large cardinal properties of small cardinals are preserved by sufficiently nice forcings.

Theorem (Foreman 2013)

Generic supercompactness of ω_1 is preserved by all proper forcings.

Sharps

Definition (Silver et al.)

$0^\#$ exists if (equivalently) each of the following objects exist:

1. An uncountable set of ordinals which are order-indiscernible over L.
2. A non-trivial elementary embedding $j : L \to L$.
3. A countable structure $(L_\alpha, \in, U)$ such that
   - $(L_\alpha, \in)$ is a model of ZFC^- with a largest cardinal κ,
   - $(L_\alpha, \in, U) \models \Sigma_0$-separation + U is a $<\kappa$-complete ultrafilter on κ
   - All iterated ultrapowers of $(L_\alpha, \in, U)$ are wellfounded.

   The least such structure is denoted $M_0^\#$.

More generally, $x^\#$ is defined for any real x by replacing L with $L[x]$.

A measurable cardinal implies that $x^\#$ exists for all reals x.

Analytic determinacy and sharps

Theorem (Martin 1970, Harrington 1978)

The following conditions are equivalent:

1. $x^{\#}$ *exists for all reals* x
2. *Analytic determinacy*

Cohen forcing

Proposition (folklore?)

Cohen forcing preserves analytic determinacy.

Proof.

Suppose:

- $x^\#$ exists for all reals x.
- $V[G]$ is a Cohen extension of V. Let x denote the Cohen real.
- σ is a name for a new real. We can assume σ is a nice name.

The name σ is essentially a real, since Cohen forcing has the c.c.c. Thus $\sigma^\#$ exists. Hence there is a nontrivial elementary embedding $j\colon L[\sigma] \to L[\sigma]$.

> *x is Cohen generic over $L[\sigma]$, since Cohen forcing has the c.c.c. and is Σ^1_2-definable.*

We can lift j to $j^* : L[\sigma][G] \to L[\sigma][G]$ as in the Levy-Solovay theorem. Since the new real $\sigma^G \in L[\sigma][G]$, this yields $(\sigma^G)^\#$ in $V[x]$. □

Sacks forcing

Proposition (Castiblanco, S. 2020)

Sacks forcing $\mathbb{P}$ preserves analytic determinacy.

Proof sketch.

Again, we obtain a small $\mathbb{P}$-name σ and a nontrivial elementary embedding $j\colon L[\sigma] \to L[\sigma]$.

> *Force over $L[\sigma]$ in V with finite subtrees of $2^{<\omega}$ ordered by end extension.*

This adds a perfect tree T such that all its branches are Cohen reals over $L[\sigma]$. This remains true in generic extensions of V.

Force with $\mathbb{P}$ below $T \in \mathbb{P}$ over V. Let x denote the Sacks real.

> *Then x is Cohen generic over $L[\sigma]$.*

Again, we can lift j to $j^*\colon L[\sigma][x] \to L[\sigma][x]$ and obtain $(\sigma^x)^\#$ in $V[x]$. $\square$

Definition

1. An operator is a function $\mathbb{M}$ that sends each real x to a structure $\mathbb{M}(x) = (M(x), \in, E)$ such that
   - $x \in M(x)$, $M(x)$ is transitive, $M(x) \models \mathrm{ZFC}^-$ and E is $M(x)$-amenable.
2. $\mathbb{M}$ is called $\mathbb{Q}$-amenable if each $\mathbb{M}(x)$ is $\mathbb{Q}$-*amenable*.

For example, let $\mathbb{M}(x) = (L(x), \in)$.

Definition (Castiblanco, S. 2020)

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcings and M is a $\mathbb{Q}$-amenable operator. We say that $\mathbb{P}$ is captured by $\mathbb{Q}$ over M if the following holds for any $p \in \mathbb{P}$ and any $\mathbb{P}$-name τ for a real: for any real x, there exists a real y with $x \in L(y)$ such that some $q \leq_{\mathbb{P}} p$ forces:

"There exists a $\mathbb{Q} \cap M(y)$-generic filter g over $M(y)$ with $\tau \in M(y)[g]$."

In other words, if H is a $\mathbb{P}$-generic filter over V, then τ^H is contained in a $\mathbb{Q}$-generic extension of some $M(y)$.

Capturing

Capturing over L implies preservation of analytic determinacy, for proper forcings on the reals.

All proper-for-candidates Suslin forcings are captured over L. This includes most classical proper forcings which add a real.

Iterable structures

One can reformulate the above preservation proofs using iterable structures $(M, \in, U)$ instead of elementary embeddings $j\colon L(\sigma) \to L(\sigma)$.

The Cohen or Sacks real x over V is generic over some $(M_0, \in, U)$. The iteration lifts step by step to

$$M_0[x] \to M_1[x] \to \cdots \to M_\alpha[x] \to \ldots$$

Iterable structures

More generally, work with transitive structures $(M, \in, E)$ where

- $(M, \in) \models \mathrm{ZFC}^-$
- E is an M-amenable sequence of (partial) extenders
- All extenders in E, except possibly the last one, are elements of M

A extender is a directed system of ultrafilters.

Definition

We call $(M, \in, E)$ ω_1-iterable if all ultrapowers in countable iteration trees (Martin, Steel) on M using E, for some strategy choosing branches, are wellfounded.

Definition

We call $(M, \in, E)$ n-tall if M has n Woodin cardinals and a measurable cardinal above them, witnessed by E. [2]

An operator $\mathbb{M}$ is called n-tall, ..., if each $\mathbb{M}(x)$ is n-tall, ...

[2] In $(M, \in, U)$, $\mathrm{crit}(U)$ is considered measurable.

A characterisation of PD

One may use the following reformulation of PD.

Theorem (Harrington, Martin, Steel, Woodin, Neeman)
The following statements are equivalent:

1. *There exists an n-tall* ω_1*-iterable operator* $\mathbb{M}$.
2. $\mathbf{\Pi}^1_{n+1}$*-determinacy.*

Preservation of PD

The operator $\mathbb{M}$ often suffices to iterate $\mathbb{M}$ in the following sense.

Definition (Schilhan, Schürz, S. 2022)

Suppose that $\mathbb{M}$ and $\mathbb{N}$ are operators. $\mathbb{M}$ is called $\mathbb{N}$-iterable if there is a term t such that for any real x and any countable iteration tree T on $\mathbb{M}(x)$:

1. *(Existence)* For any real y with $T \in \mathbb{N}(y)$, $t^{\mathbb{N}(y)}$ is a wellfounded branch in T.
2. *(Uniqueness)* For all reals y and z with $T \in \mathbb{N}(y)$ and $T \in \mathbb{N}(z)$, $t^{\mathbb{N}(y)} = t^{\mathbb{N}(z)}$.

Definition (Schilhan, Schürz, S. 2022)

Suppose that $\mathbb{M}$ and $\mathbb{N}$ are operators. $\mathbb{M}$ is called stably $\mathbb{N}$-iterable if there is a term t such that for any countable iteration tree T on a small generic extension $\mathbb{M}(x)[g]$:

1. *(Existence)* For any small generic extension $\mathbb{N}(y)[h]$ with $T \in \mathbb{N}(y)[h]$, $t^{\mathbb{N}(y)[h]}$ is a wellfounded branch in T.
2. *(Uniqueness)* For all small generic extensions $\mathbb{N}(y)[h]$ and $\mathbb{N}(z)[i]$ with $T \in \mathbb{N}(y)[h]$ and $T \in \mathbb{N}(z)[i]$, $t^{\mathbb{N}(y)[h]} = t^{\mathbb{N}(z)[i]}$.

Preservation of PD

In the above characterisation of Π^1_{n+1}-determinacy, we can add that $\mathbb{M}$ is (stably) $\mathbb{M}$-iterable, since one can show

- the $\mathbb{M}_n^\#$-operator from inner model theory is stably $\mathbb{M}_k^\#$-iterable for any $k \geq n-1$.

Proposition (Schilhan, Schürz, S. 2022)
Suppose that $\mathbb{M}$ is any stably $\mathbb{M}$-iterable $\mathbb{P}$-amenable operator and $\mathbb{P}$ is captured over $\mathbb{M}$.

1. In any $\mathbb{P}$-generic extension, $\mathbb{M}$ can be extended to an $\mathbb{M}^*$-iterable operator $\mathbb{M}^*$.
2. If $\mathbb{M}$ is n-tall, then $\mathbb{M}^*$ is n-tall.

Main result

Theorem (Schilhan, Schürz, S. 2021)
Let $\mathbb{M}$ be an operator such that ω_1 is inaccessible in each $\mathbb{M}(x)$. Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha : \alpha < \kappa \rangle$ is a countable support iteration of proper-for-candidate Suslin forcing notions $\mathbb{P}_\alpha$ such that for every $\alpha < \kappa$,

> $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{P}}_\alpha$ *is proper-for-candidates in every small generic extension of any $\mathbb{M}(x)$.*

Then $\mathbb{P}$ is captured over $\mathbb{M}$ by a countable support iteration of Suslin proper-for-candidates forcings of countable length.

Future directions

The results immediately imply projective absoluteness.

It is natural to aim for stronger determinacy principles using other operators.

We have results about (not) adding classes to equivalence relations by iterated forcing. However, at the moment this only works for c.c.c. forcings and Sacks forcing.

To my knowledge, it is open whether proper implies proper-for-candidates for Borel forcings, possibly assuming analytic determinacy.