Forcing axioms via ground model interpretations

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Motivation: Martin's axiom, PFA

MA_{ω_1}, Martin's axiom at ω_1 (Martin and Solovay 1970) was introduced to axiomatise models obtained by iterated c.c.c. forcing constructions.

- It solves problems about properties of null sets, the size of $2^{\aleph_0}$ and others.

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PFA, the proper forcing axiom, was introduced in the 1970s.

- Baumgartner used it to settle many questions left open by MA_{ω_1}. For instance, he showed that any two $\aleph_1$-dense sets of reals are isomorphic and $\square_{\omega_1}$ fails.

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Theorem (Baumgartner 1984)
Suppose that PFA^+ holds and $\kappa > \omega_1$ is regular. If $S \subseteq \kappa$ is stationary in cofinality ω, then $S \cap \alpha$ is stationary in α for some $\alpha < \kappa$ of uncountable cofinality.

Definition

Suppose that $\mathbb{P}$ is a forcing and κ is an uncountable cardinal. The forcing axiom $\mathsf{FA}_{\mathbb{P},\kappa}$ says:

> *Whenever $\vec{D} = \langle D_\alpha \mid \alpha < \kappa \rangle$ is a sequence of predense subsets of $\mathbb{P}$, there is a filter g on $\mathbb{P}$ such that $g \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.*

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Thus $MA_{\omega_1} = FA_{\mathrm{c.c.c.},\omega_1}$ and $PFA = FA_{\mathrm{proper},\omega_1}$.

Definition

The forcing axiom $FA^+_{\mathbb{P},\kappa}$ says:

> *Suppose $\vec{D} = \langle D_\alpha : \alpha < \kappa \rangle$ is a sequence of dense subsets of $\mathbb{P}$ and σ is a nice name for a subset of κ such that $\Vdash_{\mathbb{P}}$ "σ is stationary". Then there is a filter g such that*
>
> 1. *For all α, $g \cap D_\alpha \neq \emptyset$; and*
> 2. *σ^g is stationary.*

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- $MM \not\Rightarrow PFA^+$ (Shelah 1987)

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Definition

Suppose that κ is an uncountable cardinal. The name principle $\mathsf{N}_{\mathbb{P},\kappa}$ says:

> *"Whenever σ is a nice name for a subset of κ and A is a subset of κ such that $\Vdash_{\mathbb{P}} \sigma = \check{A}$, then there is a filter $g \in V$ such that $\sigma^g = A$."*

Name principles: key definition II

One can replace the formula "$x = \check{A}$" in the previous definition by any other formula $\varphi(x)$ and thus define φ-$\mathsf{N}_{\mathbb{P},\kappa}$.

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Definition

Suppose that κ is an uncountable cardinal. The simultaneous Σ_0-name principle $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa}$ says:

> *"Whenever $\sigma_0, \ldots, \sigma_n$ are nice names for subsets of κ, there is a filter g in V such that $\varphi(\sigma_0^g, \ldots, \sigma_n^g)$ holds for every Σ_0-formula φ such that $\Vdash_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_n)$."*

Example

$\mathrm{FA}_{\mathbb{P},\omega_1}$ implies that $\mathbb{P}$ does not force that any given stationary subset S of ω_1 is destroyed. Why does this follow from the Σ_0-name principle?

Towards a contradiction, suppose there is a name τ for a club with $\Vdash_{\mathbb{P}} \tau \cap S = \emptyset$. By the Σ_0-name principle, there is a filter $g \in V$ such that τ^g is club and $\tau^g \cap S = \emptyset$. But the existence of τ^g contradicts the assumption that S is stationary.

The main result, simplified

Theorem

Suppose that $\mathbb{P}$ is a forcing and κ is an uncountable cardinal. Then the following statements are equivalent:

1. $\mathsf{FA}_{\mathbb{P},\kappa}$
2. *The name principle $\mathsf{N}_{\mathbb{P},\kappa}$ for the formula $\sigma = \kappa$, where σ is any nice name for a subset of κ.*
3. *The simultaneous name principle $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa}$ for all first-order formulas over the structure $(\kappa, \in, \sigma)$, where σ is any nice name for a subset of κ.*

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The main results are more general and cover:

(i) Names of arbitrary ranks instead of nice (i.e. rank 1) names

(ii) Bounded forcing axioms and bounded name principles

A proof sketch

Lemma

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Proof.

$\Leftarrow$: Let $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of predense sets in $\mathbb{P}$. Let

$$\sigma = \{\langle \check{\alpha}, p \rangle : \alpha < \omega_1,\ p \in D_\alpha\}$$

Then $\Vdash_{\mathbb{P}} \sigma = \omega_1$. $\square$

Lemma

$\mathsf{FA}_{\mathbb{P},\omega_1} \iff \mathsf{N}_{\mathbb{P},\omega_1}$.

Proof.

$\Rightarrow$: Suppose σ is a nice name for a subset of ω_1 and $\Vdash_{\mathbb{P}} \sigma = \check{A}$.

We want a filter g with $\sigma^g = A$. Note that $\sigma^g \subseteq A$ holds for any filter g on $\mathbb{P}$, since σ is a nice name.

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For each $\alpha \in A$,

$$D_\alpha = \{p \in \mathbb{P} : \langle \check{\alpha}, p \rangle \in \sigma\}$$

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Let g meet D_α for all $\alpha \in A$. Then for all $\alpha \in A$, $\alpha = \check{\alpha}^g \in \sigma^g$. $\square$

Applications I

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Assuming CH, we have

$$\mathsf{SCFA} \iff \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathrm{subcomplete},X,\omega_1}(2) \iff \mathsf{N}_{\mathrm{subcomplete},X,\omega_1}(2)$$

for some transitive $X \in H_{\omega_1}$.

This means rank 2 names σ for ground model subsets of X, i.e. such that the names appearing in σ are nice names, can be interpreted correctly.

Applications II

Theorem

Suppose that κ is an uncountable cardinal and $\mathbb{P}$ is a forcing. Then conditions 1, 2, 3 are equivalent:

1. $\mathsf{BFA}_{\mathbb{P},\kappa}$
2. $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}_{\mathbb{P},\kappa}$
3. $\Vdash_{\mathbb{P}} V \prec_{\Sigma^1_1(\kappa)} V[\dot{G}]$

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If $\mathrm{cof}(\kappa) > \omega$*, or* $\mathrm{cof}(\kappa) = \omega$ *and there exists no inner model with a Woodin cardinal, then these are also equivalent to 4:*

4. $\Vdash_{\mathbb{P}} H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[\dot{G}]}_{\kappa^+}$

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4. $\Vdash_{\mathbb{P}} H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[\dot{G}]}_{\kappa^+}$

If $\mathrm{cof}(\kappa) = \omega$ *and* $2^{<\kappa} = \kappa$, *then these are equivalent to 5:*

5. $1_{\mathbb{P}}$ *forces that there are no new bounded subset of* κ *in* $V[\dot{G}]$

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New forcing axioms

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To this end, it is useful to study new forcing axioms such as:

Definition

Let κ be a cardinal. The unbounded forcing axiom $\text{ub-FA}_{\mathbb{P},\kappa}$ says:

> *"If $\langle D_\gamma : \gamma < \kappa \rangle$ is a sequence of κ many predense sets, then there is a filter $g \in V$ which meets unboundedly many D_γ."*

Problem

Under which conditions on $\mathbb{P}$ *does* $\text{ub-FA}_{\mathbb{P},\omega_1} \implies \text{FA}_{\mathbb{P},\omega_1}$ *hold?*

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Observation

For any σ-distributive forcing $\mathbb{P}$, ub-FA$_{\mathbb{P},\omega_1} \implies$ FA$_{\mathbb{P},\omega_1}$.

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Observation

For any σ-distributive forcing $\mathbb{P}$, ub-FA$_{\mathbb{P},\omega_1} \implies$ FA$_{\mathbb{P},\omega_1}$.

An application of the previous theorem:

Corollary

If $\mathbb{P}$ is a complete Boolean algebra that does not add reals, then

$$(\forall q \in \mathbb{P} \text{ ub-FA}_{\mathbb{P}_q,\omega_1}) \implies \text{BFA}^{\omega_1}_{\mathbb{P},\omega_1}.$$

New forcing axioms

If $\mathbb{P}$ adds reals, then the implication $\text{ub-FA}_{\mathbb{P},\omega_1} \implies \text{FA}_{\mathbb{P},\omega_1}$ may or may not hold:

$\text{ub-FA}_{\mathbb{P},\omega_1}$ holds for Cohen forcing and in fact for all σ-centred forcings. But $\text{FA}_{\text{Cohen},\omega_1}$ implies $\neg$CH.

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Proposition

Let $\mathbb{Q}$ be random forcing. The following are equivalent:

1. $\text{FA}_{\mathbb{Q},\omega_1}$
2. $\text{ub-FA}_{\mathbb{Q},\omega_1}$
3. 2^ω is not the union of ω_1 many null sets

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We also don't know if $\text{ub-FA}_{\mathbb{P},\kappa}$ always implies $\text{stat-FA}_{\mathbb{P},\kappa}$. There are numerous interesting open questions in our preprint!