Effective overtness for generalised Cantor spaces

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Dagstuhl, 17. November 2021

Generalised descriptive set theory

Suppose that κ is a regular uncountable cardinal with $\kappa^{<\kappa} = \kappa$.

The generalised Cantor space 2^κ consists of all functions $f\colon \kappa \to \{0,1\}$. Its basic open sets are

$$N_t = \{x \in 2^\kappa \mid t \subseteq x\}$$

for $t \in 2^{<\kappa}$.

There is active research on definable subsets, isomorphism relations for structures of size κ and many other topics.

Motivation

Galeotti and Nobrega (2017) investigated the computational strength of the intermediate value theorem IVT_κ for a real closed field $\mathbb{R}_\kappa$ of size 2^κ in the Weihrauch lattice via computability on generalised Cantor spaces, analogous to Brattka and Gherardi (2011).

Ingredients:

- A notion of computable function $f\colon 2^\kappa \to 2^\kappa$.
- A notion of represented space via 2^κ.

Computable functions on 2^κ

A function $f\colon 2^\kappa \to 2^\kappa$ is κ-computable if (equivalently):

1. (set theory) $f(x)$ is uniformly Δ_1-definable over $L_\kappa[x]$.

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1. (set theory) $f(x)$ is uniformly Δ_1-definable over $L_\kappa[x]$.
2. (computability) $f(x)$ is computed bit by bit in time κ by a (type 2) computation with input x using a κ-Turing machine with:
   - A finite set of states with a transition table
   - A finite number of input tapes, a finite number of work tapes and a single output tape. The tapes have length κ and entries in $\{0, 1\}$
   - A fixed limit rule: take the lim inf of the tape contents in each cell at limit times

Spaces represented by 2^κ

A κ-represented space X is given by a partial surjection $f :\subseteq 2^\kappa \to X$.

The set $\mathcal{O}(X)$ of open subsets of a κ-represented space can be defined as $\mathcal{C}(X, \mathbb{S})$, where $\mathbb{S}$ is the Sierpinski space.

$\mathcal{O}(X)$ is itself a κ-represented space.

Overtness

Galeotti and Nobrega used a κ-computable function $f\colon \kappa \to 2^\kappa$ with a dense image, but observed that such a function exists if and only if $P(\lambda) \subseteq L$ for all $\lambda < \kappa$.

Pauly asked whether a weaker condition – overtness – holds for generalised Cantor spaces (as represented spaces).

Overtness

The notion of overtness is dual to compactness. It is trivially true for all topological spaces, but it does not in general hold effectively for represented spaces.

Definition

A κ-represented space is called effectively κ-overt if for any κ-represented space Y, the projection

$$p\colon \mathcal{O}(X \times Y) \to \mathcal{O}(X), \quad p(U) = \{x \in X \mid \exists y \in Y\ (x, y) \in U\}$$

is well-defined and κ-computable.

Is 2^κ effectively overt?

Theorem (Pauly, S.)

1. 2^{ω_1} *is effectively overt.*
2. *It is consistent that* $V \neq L$ *and* 2^κ *is effectively overt for any regular cardinal* $\kappa \geq \omega_2$.
3. 2^κ *is not effectively overt for all regular cardinals* $\kappa \geq \omega_2$*, assuming:*
    - V *is obtained by adding a Cohen real to* L*, or*
    - $0^\#$ *exists.*

Questions

- Do you know of applications of overtness that might follow from the above at ω_1?
- Which results would you hope for in a synthetic topology of generalised Cantor spaces?