Forcing axioms via ground model interpretations

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Motivation: Martin's axiom, PFA

MA_{ω_1}, Martin's axiom at ω_1 (independently Martin, Kunen, Rowbottom, Tennenbaum 1970) was introduced to axiomatise models obtained by iterated c.c.c. forcing constructions.

- It solves problems about properties of null sets, the size of $2^{\aleph_0}$ and others.

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- It solves problems about properties of null sets, the size of $2^{\aleph_0}$ and others.

PFA, the proper forcing axiom, was introduced in the 1970s by Shelah.

- Baumgartner used it to settle many questions left open by MA_{ω_1}. For instance, he showed that any two $\aleph_1$-dense sets of reals are isomorphic. Todorčević showed that $\square_{\omega_1}$ fails.

Motivation: Strong forcing axioms

Baumgartner (1984) introduced stronger forcing axioms $MA^{+}_{\omega_1}$ and PFA^{+}. (His terminology was slightly different.)
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He used it to prove stationary reflection.

Theorem (Baumgartner 1984)
Suppose that PFA^+ holds and $\kappa > \omega_1$ is regular. If $S \subseteq \kappa$ is stationary in cofinality ω, then $S \cap \alpha$ is stationary in α for some $\alpha < \kappa$ of uncountable cofinality.

Definition

Suppose that $\mathbb{P}$ is a forcing and κ is an uncountable cardinal. The forcing axiom $\mathsf{FA}_{\mathbb{P},\kappa}$ says:

> *Whenever $\vec{D} = \langle D_\alpha \mid \alpha < \kappa \rangle$ is a sequence of predense subsets of $\mathbb{P}$, there is a filter g on $\mathbb{P}$ such that $g \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.*

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Thus $MA_{\omega_1} = FA_{\mathrm{c.c.c.},\omega_1}$ and $PFA = FA_{\mathrm{proper},\omega_1}$.

Definition

The forcing axiom $FA^{+}_{\mathbb{P},\kappa}$ says:

> *Suppose* $\vec{D} = \langle D_\alpha : \alpha < \kappa \rangle$ *is a sequence of dense subsets of* $\mathbb{P}$ *and* σ *is a nice name for a subset of* κ *such that* $\Vdash_{\mathbb{P}}$ *"*σ *is stationary". Then there is a filter* g *such that*
>
> 1. *For all* α, $g \cap D_\alpha \neq \emptyset$; *and*
> 2. σ^g *is stationary.*

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- $FA_{\sigma-closed} \not\Rightarrow FA^{+}_{\sigma-closed}$ (Baumgartner 1984)

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- $PFA \not\Rightarrow PFA^+$ (Beaudoin 1987, Magidor 1987) via the failure of stationary reflection
- $MM \not\Rightarrow PFA^+$ (Shelah 1987)

Name principles: key definition I

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Definition

Suppose that κ is an uncountable cardinal. The name principle $\mathsf{N}_{\mathbb{P},\kappa}$ says:

> *"Whenever σ is a nice name for a subset of κ and A is a subset of κ such that $\Vdash_{\mathbb{P}} \sigma = \check{A}$, then there is a filter $g \in V$ such that $\sigma^g = A$."*

Name principles: key definition II

One can replace the formula "$x = \check{A}$" in the previous definition by any other formula $\varphi(x)$ and thus define φ-$\mathsf{N}_{\mathbb{P},\kappa}$.

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Definition

Suppose that κ is an uncountable cardinal. The simultaneous Σ_0-name principle $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa}$ says:

> *"Whenever $\sigma_0, \ldots, \sigma_n$ are nice names for subsets of κ, there is a filter g in V such that $\varphi(\sigma_0^g, \ldots, \sigma_n^g)$ holds for every Σ_0-formula φ such that $\Vdash_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_n)$."*

Example

$\mathsf{FA}_{\mathbb{P},\omega_1}$ implies that $\mathbb{P}$ does not force that any given stationary subset S of ω_1 is destroyed. Why does this follow from the Σ_0-name principle?

Towards a contradiction, suppose there is a name τ for a club with $\Vdash_{\mathbb{P}} \tau \cap S = \emptyset$. By the Σ_0-name principle, there is a filter $g \in V$ such that τ^g is club and $\tau^g \cap S = \emptyset$. But the existence of τ^g contradicts the assumption that S is stationary.

Theorem

Suppose that $\mathbb{P}$ is a forcing and κ is an uncountable cardinal. Then the following statements are equivalent:

1. $\mathsf{FA}_{\mathbb{P},\kappa}$
2. *The name principle $\mathsf{N}_{\mathbb{P},\kappa}$ for the formula $\sigma = \kappa$, where σ is any nice name for a subset of κ.*
3. *The simultaneous name principle $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},\kappa}$ for all first-order formulas over the structure $(\kappa, \in, \sigma)$, where σ is any nice name for a subset of κ.*

The correspondence, simplified

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A proof sketch

Lemma

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Proof.

$\Leftarrow$: Let $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of predense sets in $\mathbb{P}$. Let

$$\sigma = \{\langle \check{\alpha}, p \rangle : \alpha < \omega_1,\ p \in D_\alpha\}$$

Then $\Vdash_{\mathbb{P}} \sigma = \omega_1$. $\square$

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$\mathsf{FA}_{\mathbb{P},\omega_1} \iff \mathsf{N}_{\mathbb{P},\omega_1}$.

Proof.

$\Rightarrow$: Suppose σ is a nice name for a subset of ω_1 and $\Vdash_{\mathbb{P}} \sigma = \check{A}$.

We want a filter g with $\sigma^g = A$. Note that $\sigma^g \subseteq A$ holds for any filter g on $\mathbb{P}$, since σ is a nice name.

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Proof.

$\Rightarrow$: Suppose σ is a nice name for a subset of ω_1 and $\Vdash_{\mathbb{P}} \sigma = \check{A}$.

We want a filter g with $\sigma^g = A$. Note that $\sigma^g \subseteq A$ holds for any filter g on $\mathbb{P}$, since σ is a nice name.

For each $\alpha \in A$,

$$D_\alpha = \{p \in \mathbb{P} : \langle \check{\alpha}, p \rangle \in \sigma\}$$

is predense since $\Vdash_{\mathbb{P}} \sigma = \check{A}$.

Let g meet D_α for all $\alpha \in A$. Then for all $\alpha \in A$, $\alpha = \check{\alpha}^g \in \sigma^g$. $\square$

A proof sketch

Lemma

$\mathsf{FA}_{\mathbb{P},\omega_1} \iff \Sigma_0^{\text{(sim)}}\text{-}\mathsf{N}_{\mathbb{P},\kappa}$.

Proof.

$\Leftarrow$: By the previous lemma, since $\mathsf{N}_{\mathbb{P},\kappa}$ follows from $\Sigma_0^{\text{(sim)}}\text{-}\mathsf{N}_{\mathbb{P},\kappa}$ as a special case for the formula $\sigma = \check{A}$.

$\Rightarrow$: By induction on formulas as described below. $\square$

Definition

A name σ has (name) rank α if either

1. $\alpha = 0$ and $\sigma = \check{x}$ for some x, or
2. $\alpha > 0$ is least such that every name in σ has rank $<\alpha$.

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Definition

A name σ is *locally κ-small* if there are at most κ many names τ such that for some $p \in \mathbb{P}$, $\langle \tau, p \rangle \in \sigma$.

σ is *κ-small* if it has rank 0, or it is locally κ-small and all the names it contains are κ-small.

Rank 1 names for subsets of κ are always small.

Definition

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1. $\alpha = 0$ and $\sigma = \check{x}$ for some x, or
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Definition

A name σ is *locally λ-bounded* if for every name τ, there are at most λ many $p \in \mathbb{P}$ such that $\langle \tau, p \rangle \in \sigma$.

σ is *λ-bounded* if it is rank 0, or is locally λ-bounded and all the names it contains are λ-bounded.

Name principles for arbitrary ranks

Suppose that $\alpha \in \mathrm{Ord} \cup \{\infty\}$ and X is a set of size $\leq\kappa$.

Definition

The name principle $\mathsf{N}_{\mathbb{P},\kappa}(\alpha)$ states:
If σ is a κ-small name of rank $\leq \alpha$ and $A \in H_{\kappa^+} \cap P^\alpha(X)$ is such that $\mathbb{P} \Vdash \sigma = \check{A}$, then there is a filter $g \in V$ with $\sigma^g = A$.

Note: The requirement $A \in H_{\kappa^+}$ is necessary, since for some $\mathbb{P}$ there are ω-bounded rank 2 names $\sigma \in H_{\omega_1}$ for $P(\omega)^V$.

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Definition

The first order name principle $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{N}_{\mathbb{P},\kappa}(\alpha)$ states:
If $\sigma_1, \ldots, \sigma_n$ are κ-small names of rank $\leq\alpha$ and $\varphi(v_1, \ldots, v_n)$ is any Σ_0 formula such that $\mathbb{P} \Vdash \varphi(\vec{\sigma})$, then there is a filter $g \in V$ such that $V \models \varphi(\vec{\sigma}^g)$.

Theorem

Let $\mathbb{P}$ be a forcing and let κ be a cardinal. The following implications hold, given the assumptions noted at the arrows:

1.

$$\begin{tikzcd} & \mathsf{FA}_\kappa \arrow[dr, bend left] & \\ \mathsf{N}_{\mathbb{P},\kappa}(\infty) \arrow[ur, bend left] & & \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},\kappa}(\infty) \arrow[ll] \end{tikzcd}$$

A similar result holds for the λ-bounded versions, where $\lambda \geq \kappa$ is a cardinal.

The correspondence

Theorem

Let $\mathbb{P}$ be a forcing and let κ be a cardinal. The following implications hold, given the assumptions noted at the arrows:

2. *For any ordinal $\alpha > 0$, and any transitive set X of size at most κ:*

$$\begin{tikzcd} & \mathsf{FA}_\kappa \arrow[dr, bend left] & \\ \mathsf{N}_{\mathbb{P},X,\kappa}(\alpha) \arrow[ur, bend left, "|\mathcal{P}^{<\alpha}(X)|\geq\kappa"] & & \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathbb{P},X,\kappa}(\alpha) \arrow[ll] \end{tikzcd}$$

A similar result holds for the λ-bounded versions, where $\lambda \geq \kappa$ is a cardinal.

A proof sketch

Lemma

Let $\vec{\sigma}$ be a finite tuple of κ-small names and let φ be Σ_0. Then there is a collection $\mathcal{D} = \mathcal{D}_{\varphi(\vec{\sigma})}$ of at most κ many dense sets, such that if g is any filter and

1. *g meets every element of $\mathcal{D}$*
2. *g contains some p such that $p \Vdash \varphi(\vec{\sigma})$*

then $V \vDash \varphi(\vec{\sigma}^g)$.

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then $V \vDash \varphi(\vec{\sigma}^g)$.

We prove the lemma for the following cases in turn:

1. φ is of the form $\sigma = \check{A}$
2. φ is of the form $\tau = \sigma$
3. φ is of the form $\tau \in \sigma$
4. φ is a negation of one of the above three forms
5. Arbitrary φ

φ is of the form $\sigma = \check{A}$

Proof ($\sigma = \check{A}$ case, sketch).
Induction on the rank of σ.

Write the elements of σ as $\tau_\gamma : \gamma < \kappa$.

For $B \in A$, define a dense set D_B to ensure that B ends up in σ^g:

$$D_B = \left\{ p \in \mathbb{P} : p \Vdash \sigma \neq \check{A} \vee \exists\gamma\big(p \Vdash^+ \tau_\gamma \in \sigma \wedge p \Vdash \tau_\gamma = \check{B}\big) \right\}$$

For $\gamma < \kappa$, define a dense set E_γ in a similar way. Let

$$\mathcal{D}_{\sigma=\check{A}} = \{D_B : B \in A\} \cup \{E_\gamma : \gamma < \kappa\} \cup \bigcup_{B \in A, \gamma < \kappa} \mathcal{D}_{\tau_\gamma = \check{B}}$$

□

Applications I

We can characterize PFA as follows:

$$\mathsf{PFA} \iff \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathrm{proper},\omega_1} \iff \mathsf{N}_{\mathrm{proper},\omega_1}.$$

In other words, rank 1 names for ω_1 can be interpreted correctly.

$$\mathsf{PFA} \iff \Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{N}_{\mathrm{proper},\omega,\omega_1}(2) \iff \mathsf{N}_{\mathrm{proper},\omega,\omega_1}(2).$$

So rank 2 names for sets of reals can be interpreted correctly.

Previous results on generic absoluteness

Theorem (Bagaria 2000)

Let $\mathbb{P}$ be a partial ordering and κ an infinite cardinal of uncountable cofinality. Then the following are equivalent:

1. $\mathrm{BFA}_\kappa(\mathbb{P})$
2. $\Sigma_1(H_{\kappa^+})$*-absoluteness for* $\mathbb{P}$.

This builds on a previous result of Bagaria (1997).

Before, Fuchino had characterised Martin's axiom by the existence of embeddings (1992).

Theorem

Suppose that κ is an uncountable cardinal and $\mathbb{P}$ is a forcing. Then conditions 1, 2, 3 are equivalent:

1. $\mathsf{BFA}_{\mathbb{P},\kappa}$
2. $\Sigma_0^{(\mathrm{sim})}$-$\mathsf{BN}_{\mathbb{P},\kappa}$
3. $\Vdash_{\mathbb{P}} V \prec_{\Sigma^1_1(\kappa)} V[\dot{G}]$

Applications II

Theorem
Suppose that κ is an uncountable cardinal and $\mathbb{P}$ is a forcing. Then conditions 1, 2, 3 are equivalent:

1. $\mathsf{BFA}_{\mathbb{P},\kappa}$
2. $\Sigma_0^{(\mathrm{sim})}\text{-}\mathsf{BN}_{\mathbb{P},\kappa}$
3. $\Vdash_{\mathbb{P}} V \prec_{\Sigma^1_1(\kappa)} V[\dot{G}]$

If $\mathrm{cof}(\kappa) > \omega$, *or* $\mathrm{cof}(\kappa) = \omega$ *and there exists no inner model with a Woodin cardinal, then these are also equivalent to 4:*

4. $\Vdash_{\mathbb{P}} H^V_{\kappa^+} \prec_{\Sigma_1} H^{V[\dot{G}]}_{\kappa^+}$

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If $\mathrm{cof}(\kappa) = \omega$ *and* $2^{<\kappa} = \kappa$, *then these are equivalent to 5:*

5. $1_{\mathbb{P}}$ *forces that there are no new bounded subset of* κ *in* $V[\dot{G}]$

Theorem

The following statements are equivalent for Boolean ultrapower embeddings $j_U\colon V \to \check{V}_U$:

1. $\mathrm{FA}_{\mathbb{P},\kappa}$
2. *For any transitive set $M \in H_{\kappa^+}$ and for every κ-small M-name σ, there is an ultrafilter $U \in V$ on $\mathbb{P}$ such that*

$$j_U{\upharpoonright}M\colon M \to j_U(M)^{\in_U}$$

is an elementary embedding from $(M, \in, \sigma^U)$ to $(j_U(M)^{\in_U}, \in_U, [\sigma]_U)$.

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New forcing axioms

We began an analysis of various name principles for notions of largeness for subsets of κ, e.g. being unbounded or stationary.

To this end, it is useful to study new forcing axioms such as:

Definition

Let κ be a cardinal. The unbounded forcing axiom ub-$\mathsf{FA}_{\mathbb{P},\kappa}$ says:

> *"If $\langle D_\gamma : \gamma < \kappa \rangle$ is a sequence of κ many predense sets, then there is a filter $g \in V$ which meets unboundedly many D_γ."*

Implications

$$
\begin{tikzcd}
\mathrm{N}_\kappa \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{club-}\mathrm{N}_\kappa \arrow[r, dash, dashed] \arrow[d, leftrightarrow] & \text{stat-}\mathrm{N}_\kappa \arrow[r] \arrow[d] & \text{ub-}\mathrm{N}_\kappa \arrow[d, leftrightarrow] \\
\mathrm{FA}_\kappa \arrow[r, leftrightarrow] & \text{club-}\mathrm{FA}_\kappa \arrow[r] & \text{stat-}\mathrm{FA}_\kappa \arrow[r, dotted] & \text{ub-}\mathrm{FA}_\kappa
\end{tikzcd}
$$

Solid arrows: non-reversible implications

Dotted arrows: implications whose converse remains open

Dashed lines: no implication is provable

New forcing axioms

Problem

Under which conditions on $\mathbb{P}$ does $\text{ub-FA}_{\mathbb{P},\omega_1} \implies \text{FA}_{\mathbb{P},\omega_1}$ *hold?*

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Under which conditions on $\mathbb{P}$ does ub-$\mathsf{FA}_{\mathbb{P},\omega_1} \implies \mathsf{FA}_{\mathbb{P},\omega_1}$ *hold?*

Observation

For any σ-distributive forcing $\mathbb{P}$, ub-$\mathsf{FA}_{\mathbb{P},\omega_1} \implies \mathsf{FA}_{\mathbb{P},\omega_1}$.

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Under which conditions on $\mathbb{P}$ does $\text{ub-FA}_{\mathbb{P},\omega_1} \implies \text{FA}_{\mathbb{P},\omega_1}$ *hold?*

Observation

For any σ-distributive forcing $\mathbb{P}$, $\text{ub-FA}_{\mathbb{P},\omega_1} \implies \text{FA}_{\mathbb{P},\omega_1}$.

An application of the previous theorem:

Corollary

If $\mathbb{P}$ is a complete Boolean algebra that does not add reals, then

$$(\forall q \in \mathbb{P}\ \text{ub-FA}_{\mathbb{P}_q,\omega_1}) \implies \text{BFA}^{\omega_1}_{\mathbb{P},\omega_1}.$$

Can this be extended to (ω, λ)-distributive forcings?

New forcing axioms

If $\mathbb{P}$ adds reals, then the implication $\text{ub-FA}_{\mathbb{P},\omega_1} \implies \text{FA}_{\mathbb{P},\omega_1}$ may or may not hold:

$\text{ub-FA}_{\mathbb{P},\omega_1}$ holds for Cohen forcing and in fact for all σ-centred forcings. But $\text{FA}_{\text{Cohen},\omega_1}$ implies $\neg$CH.

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Proposition

Let $\mathbb{Q}$ be random forcing. The following are equivalent:

1. $\text{FA}_{\mathbb{Q},\omega_1}$
2. $\text{ub-FA}_{\mathbb{Q},\omega_1}$
3. 2^ω is not the union of ω_1 many null sets

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We don't know if $\text{ub-FA}_{\mathbb{P},\kappa}$ always implies $\text{stat-FA}_{\mathbb{P},\kappa}$.

Stationary name principles

Lemma

If $\mathbb{P}$ is σ-distributive, then stat-$\mathrm{N}_{\mathbb{P},\omega_1}$ *implies* $\mathsf{FA}^+_{\omega_1}(\mathbb{P})$.

Theorem (Foreman, Magidor, Shelah 1988)

$\mathsf{FA}^+_{\omega_1}(\sigma\text{-}closed) \Rightarrow$ *stationary reflection for* $[\lambda]^\omega$ *for all* $\lambda \geq \omega_2$.

This has very strong consistency.

Theorem (Sakai 2014)

$\mathsf{FA}^+_{\omega_1}(Add(\omega_1, 1))$ *is nontrivial.*

Weak interpretations

Strong forcing axioms are often stated via weak interpretations

$$\sigma^{(g)} = \{\alpha < \kappa \mid \exists p \in g \ \ p \Vdash \alpha \in \sigma\}$$

of names σ for subsets of κ.

In the context of FA_κ, one does not need to distinguish between these two kinds of interpretations.

Lemma

The following are equivalent:

1. $\mathsf{FA}_{\mathbb{P},\kappa}$
2. *For every rank* 1 $\mathbb{P}$*-name* σ *for a subset of* κ*, there is a filter* g *on* $\mathbb{P}$ *with* $\sigma^{(g)} = \sigma^g$*.*

Stationary name principles

The original definition of PFA^+ combines PFA with the name principle stat-BN^1 for 1-bounded names.

(The latter is equivalent to stat-N for weak interpretations.)

Is stat-BN^1 alone nontrivial?

Stationary name principles

The next results shows that stat-$\mathrm{BN}^1_{\mathbb{P},\omega_1}$ is nontrivial.

Proposition

Let $\kappa = 2^{\aleph_0}$ and assume that $\mathrm{non}(\mathrm{null}) = 2^{\aleph_0}$. Then stat-$\mathrm{BN}^1_{\mathbb{P},\kappa}$ fails for random forcing $\mathbb{P}$. In particular, CH implies that stat-$\mathrm{BN}^1_{\mathbb{P},\omega_1}$ fails.

Proposition

Assume $\diamondsuit_{\omega_1}$. Then stat-$\mathrm{BN}^1_{T,\omega_1}$ fails for any Suslin tree T.

Lemma (with Hamkins)

Suppose $\lambda < \kappa$ and $\mathbb{P}$ is well-met. If $\text{stat-BN}^{\lambda}_{\mathbb{P},\kappa}$ fails, then there are densely many conditions $p \in \mathbb{P}$ such that $\text{stat-BN}^{1}_{\mathbb{P}_p,\kappa}$ fails, where $\mathbb{P}_p := \{q \in \mathbb{P} : q \leq p\}$.

Connections to other work

Fuchs and Minden (2018) show assuming CH: the bounded subcomplete forcing axiom BSCFA is equivalent to preservation of $(\omega_1, {\leq}\omega_1)$-Aronszajn trees *T*.
The latter is the 1-bounded name principle for statements of the form "σ is an ω_1-branch in *T*", where *T* is as above.

Bagaria's result has been extended by Fuchs (2021).
Fuchs introduced $\Sigma^1_1(\kappa, \lambda)$-absoluteness for cardinals $\lambda \geq \kappa$ and proved it is equivalent to $\mathsf{BFA}^{\lambda}_{\kappa}$. Can this be derived from our results?

Future directions

Can we separate ub-FA_{ω_1} from stat-FA_{ω_1}?
Can ub-FA_{ω_1} be nontrivial but not imply FA_{ω_1}?
Which of these hold for Baumgartner's forcing to add a club in ω_1 with finite conditions?

BPFA^+ has only been formulated as a generic absoluteness principle by artificially adding a predicate for the nonstationary ideal.
Can one formulate BPFA^+ as a generic absoluteness or name principle for a logic beyond first order?