The recognisable universe in the presence of measurable cardinals

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Overview

- Recognisable sets of ordinals were studied by Carl, Welch and Schlicht (2018)
  - They originally came from infinite time computation
  - They are connected with Hamkins' and Leahy's implicitly definable sets (2016)
- We study the recognisable universe generated by all recognisable sets
- Our aims:
  - Analyse the recognisable universe using large cardinals
  - Determine the recognisable universe in canonical inner models
- This is a joint project with Philip Welch and in part Merlin Carl

Recognisable sets

Definition

A subset of α is called recognisable if there is a first-order formula $\varphi(x)$ with ordinal parameters such that x is the unique subset of α with

$$L[x] \models \varphi(x).$$

A recognisable set need not be in L.

Example

$0^{\#}$, and all Π^1_2-singletons, are recognisable, since Π^1_2-truth is $(L[x], V)$-absolute.

Connections:

- The lost melody phenomenon: A real that is decidable, but not computable by an infinite time Turing machine
- A Π^1_1 singleton may not be Π^1_1-definable as a set of natural numbers

The recognisable universe

The class Rec of recognisable sets is not necessarily constructibly closed.

Example

A Cohen real over L cannot be recognisable. Otherwise, all Cohen reals over L extending a specific finite sequence would satisfy the formula recognising it.

But $0^{\#}$ is recognisable and constructs Cohen reals over L.

The recognisable universe R denotes the constructible closure of Rec, i.e.

$$\mathrm{R} = \bigcup_{x \in \mathrm{Rec}} L[x].$$

R equals the class of sets coded by recognisable sets, via the Mostowski collapse.

The recognisable universe

- What can we say about R?
- Is it a model of a weak set theory?
- Which large cardinals can R have?
- Does R contain certain canonical inner models?
- Can any set be put into R by forcing over V?

Connection I:
Infinite time computation

Rec can be defined via unrestricted Turing machines with ordinal time and tape.

These were introduced by Koepke (2005), generalising Hamkins' and Kidder's infinite time Turing machines with tape ω (2000).

Proposition (csw 2018)

Rec *equals the class of subsets x of some α that can recognised by an unrestricted program $p(x)$ with finitely many ordinal parameters.*

I.e. x is the unique subset of α such that $p(x)$ halts with a specified final state. (We can assume that $p(y)$ halts for all subsets y of α.)

These machines compute precisely the constructible sets. So they can detect much more than they can compute.

Connection II: Inner models built from strong logics

Replace first-order logic by a stronger logic to obtain variants of L:

- HOD arises from second-order logic (Myhill, Scott 1971)
- Chang's model $L(\mathrm{Ord}^\omega)$ arises from $\mathcal{L}_{\omega_1,\omega_1}$ (Chang 1971)
- $L[\mathrm{Card}]$
- $L[\mathrm{Cof}\ \omega]$

While HOD is too variable to achieve a complete analysis, the remaining models have been analysed, assuming large cardinals.
(Woodin 2004, Welch 2019, Magidor, Kennedy, Väänänen 2020)

In particular, their first-order theories are absolute, assuming a proper class of Woodin cardinals.

$L[\mathrm{Card}]$ is in fact a generic extension of a fine-structural inner model by hyperclass forcing (Welch 2019).

Connection II: Inner models built from strong logics

Hamkins and Leahy (2014) studied implicitly definable sets of ordinals.

Definition (Hamkins, Leahy)

Suppose that M is a class. A subset X of M is called implicitly definable over M if for some first-order formula $\varphi(.)$ with parameters in M, X is unique with

$$(M, \in, X) \models \varphi(X).$$

Let $P_{\mathrm{imp}}(M)$ denote the set of subsets of M which are implicitly definable over M.

Let $\mathrm{Imp}_0 = \emptyset$, $\mathrm{Imp}_{\alpha+1} = P_{\mathrm{imp}}(\mathrm{Imp}_\alpha)$ and $\mathrm{Imp}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{Imp}_\alpha$ for limits λ.

$\mathrm{Imp} = \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{Imp}_\alpha$ is called the implicitly definable universe.

Some questions asked by Hamkins and Leahy (2014):

- Can Imp have measurable cardinals?
- Which large cardinals are absolute to Imp?
- Can we put arbitrary sets into the Imp of a suitable forcing extension?

Connection II:
Recognisable sets $\leftrightsquigarrow$ implicitly definable sets

Let $\mathcal{L}_{\infty,0}$ denote the quantifier-free language with infinitary disjunctions and conjunctions. Its atomic formulas are of the form $\alpha \in \cdot$.

Proposition (csw 2018)

The constructible closures of the following classes are equal:

(1) *Recognisable sets*

(2) *Sets implicitly definable over* L

(3) *Sets (finite-depth)* $\mathcal{L}_{\infty,0}$*-definable over* L *as singletons.*

Sketch of (1) $\Rightarrow$ (2): If x is recognisable, show that a set A of ordinals coding $L_\alpha[x]$ is implicitly definable over L. A encodes the construction of the $L[x]$-hierarchy.

Every set implicitly definable over L is also recognisable.

$0^\#$ is recognisable, but we do not know whether it is implicitly definable over L.

Corollary

R *is a subclass of* Imp.

- Is R a proper subclass of Imp?

Aims

Our main objectives are:

- Determine which axioms of set theory hold in R
- Calculate R
  - in canonical inner models, and
  - in the presence of large cardinals

We are also interested in the consequences for Imp.

Axioms of set theory in R

Proposition

1. R *is a model of basic axioms such as pairing, union and* Δ_0*-separation.*
2. R *is closed under recursion for functions that are absolute between inner models.*
3. *The well-ordering principle holds in* R*.*
4. R *has a definable global wellorder.*
5. $\mathrm{R}^{\mathrm{R}} = \mathrm{R}$.

Question

Is R *admissible?*

R in the presence of measurable cardinals

Let $L[\mu]$ denote a model with a normal ultrafilter μ on κ, where κ is least.

Kunen showed that $L[\mu]$ is unique: if ν is a normal ultrafilter on κ in some $L[\nu]$, then $\mu = \nu$.

This is proved by taking iterated ultrapowers of both $L[\mu]$ and $L[\nu]$ up to a regular cardinal λ, so that the images of both μ and ν equal the club filter on λ, restricted to the target model. A Skolem hull argument shows $\mu = \nu$. (See Kanamori.)

One can use this to see that μ is recognisable from κ and a sufficiently large cardinal $\lambda > \kappa$.

Similar arguments work for $L[\nu_0, \ldots, \nu_n]$, where $\nu_0, \ldots \nu_n$ are normal ultrafilters on some $\lambda_0 < \cdots < \lambda_n$.

R in $L[\vec{\mu}]$

Let

$$L[\vec{\mu}]$$

denote the least inner model with a sequence $\vec{\mu} = \langle \mu_n \mid n \in \omega \rangle$ of normal ulterafilter on measurable cardinals $\vec{\kappa} = \langle \kappa_n \mid n \in \omega \rangle$.

I.e. each κ_n is least.

Lemma (csw 2018)

A recognisable set $x \in N$ cannot construct V_λ^N, where N is a transitive model of ZFC *with measurable (in N) cardinals $\vec{\lambda} = \langle \lambda_n \mid n \in \omega \rangle$ with supremum λ.*

This is because one could form an ultrapower of N which fixes the given ordinal parameters, by a result of Kunen. (See Kanamori.)

From now on, assume

$$V = L[\vec{\mu}].$$

We have $\mathrm{R} \subsetneq L[\vec{\mu}]$ by the previous lemma.

R in $L[\vec{\mu}]$

What is R precisely?

Let

$$R_n = L(H_{\kappa_n^+}).$$

Let A_n code $H_{\kappa_n^+}$ with its canonical wellorder together with μ_n. ($L[\vec{\mu}]$ has an L-like fine structure and therefore a simply definable wellorder.)

Extensions of Kunen's argument show:

$L[A_n]$ is the unique fine-structural model with measurable cardinals precisely at $\kappa_0, \ldots, \kappa_n$ and no ultrafilters beyond μ_n.

Hence $A_n \in \mathrm{R}$ for all $n \in \omega$. It follows that $V_\kappa \subseteq \mathrm{R}$. Equivalently $\bigcup_{n\in\omega} R_n \subseteq \mathrm{R}$.

Is $\mathrm{R} = \bigcup_{n\in\omega} R_n$?

R in $L[\vec{\mu}]$

Lemma

Any set $x \in \bigcup_{n\in\omega} R_n$ is definable in some R_n from sets in V_κ and uniform indiscernibles.

Proof sketch.

Let $x \in \bigcup_{n\in\omega} R_n$. By a Skolem hull argument as for $0^\#$, x is definable from A_n and A_n-indiscernibles.

It thus suffices to prove the claim for ordinals α. We can assume that α is not A_n-indiscernible for some $n \in \omega$.

As for $0^\#$, α is definable from A_n, A_n-indiscernibles $\beta_0 < \cdots < \beta_k < \alpha$ and A_n-indiscernibles $\alpha < \gamma_0 < \cdots < \gamma_l$ for some $n \in \omega$. Here $\gamma_0 < \cdots < \gamma_l$ can be chosen as uniform indiscernibles.

Now apply the inductive hypothesis to $\beta_0 < \cdots < \beta_n$. $\square$

R in $L[\vec{\mu}]$

Solovay noticed that for iterated ultrapowers $M_0, M_1, \ldots$ with a normal ultrafilter μ on λ with embeddings $\pi_{k,n}\colon M_k \to M_n$ for $k \leq n$ and $\lambda_n = \pi_{0,n}(\lambda)$, the sequence $\vec{\lambda} = \langle \lambda_n \mid n \in \omega \rangle$ is Prikry generic over $\bigcap_{n \in \omega} M_n$.

Dehornoy (1978) proved generalisations of this to longer iterated ultrapowers.

R in $L[\vec{\mu}]$

Theorem

$\mathrm{R} = \bigcup_{n\in\omega} R_n$.

Proof sketch.

Suppose that x is recognisable from ordinals that are definable in $L[A_n]$ from A_n and uniform indiscernibles $\gamma_0 < ... < \gamma_k$ by the above lemma. We can assume x is a subset of γ_k.

The idea is to iterate $\mu_{n+1}, ..., \mu_{n+k+1}$ in the gaps between the γ_i and show that x remains an element of all iterated ultrapowers.

In successor steps, the γ_i are not moved, so the recognised set remains in the target model by elementarity.

In the limit step, we konw that x is in the intersection of the models.

Dehornoy's result shows that since x in in HOD of the intersection, it is in the target model.

One also needs a lemma about commuting ultrapowers by Kunen.

□

R in $L[\vec{\mu}]$

What is Imp in $L[\vec{\mu}]$?

We saw that $\mathrm{R}^{L[\vec{\mu}]} = \bigcup_{n\in\omega} L(H_{\kappa_n})$, where $\kappa_n = \mathrm{crit}(\mu_n)$.

Note that $\vec{\mu}$ is a definable class in R, since κ_n is the n^{th} measurable cardinal in R and μ_n is the unique normal measure on κ_n.

Since Imp is a model of ZF, we have $\vec{\mu} \in \mathrm{Imp}$ and hence $\mathrm{Imp}^{L[\vec{\mu}]} = L[\vec{\mu}]$.

In particular, $\mathrm{R}^{L[\vec{\mu}]} \subsetneq \mathrm{Imp}^{L[\vec{\mu}]}$.

R in $L[\vec{\mu}]$

R is admissible below κ:

Proposition

R *satisfies* Σ_1*-collection and* Σ_1*-replacement for relations/functions with domains in* V_κ*.*

This is proved via the previous lemma about uniform indiscernibles.

R in $L[\vec{\mu}]$

R is not admissible.

Lemma

R *is not admissible.*

Proof.

Let $F\colon \lambda \to V$ list all sound mice $M \in V_\kappa$ in the order $<^*$.

One can check that F is defined by a Σ_1-recursion.

This uses a folklore lemma that iterability is absolute between transitive models of ZFC^- containing ω_1, assuming there is no inner model with a Woodin cardinal.

It also uses that for the kinds of mice considered here, such a model can see all $=^*$-equivalence classes $<^*$-below a given mouse.

Since $L[A_n] \in \mathrm{ran}(F)$ for all $n \in \omega$, admissibility of R would imply that $\mathrm{ran}(F) \in$ R. But $\mathrm{ran}(F)$ is not contained in $L[A_n]$ for any $n \in \omega$. $\square$

R in the presence of large cardinals

Let M_1 denote the least fine-structural iterable inner model with a Woodin cardinal.

Let M^∞ denote the Ord-iterated ultrapower of M_1 by the unique normal measure on its least measurable cardinal.

Theorem (csw 2018)

All recognisable subsets of countable ordinals are elements of M^∞.

There is a converse:

Theorem (csw 2018)

Proper class many many initial segments of M^∞ are recognisable.

The problem is not fully solved for subsets of ω_1.

Theorem (csw 2018)

It is consistent that all recognisable subsets of ω_1 are in M^∞.

Here is another partial result towards this problem.

Proposition

Assume that H_{ω_2} is closed under the $M_1^{\#}$-operator.

Then every subset of ω_1 that is recognisable from countable ordinals and V-cardinals is an element of M^{∞}.

Some open questions

The following is our main question:

Question

Is R *generically absolute, assuming sufficient large cardinals?*

While every recognisable subset of a countable ordinal is in M^∞, this is not clear for Imp:

Question

Does Imp *contain* M_1*?*

Recognisable sets are connected with unique generics. The following is open:

Question

Assuming that $0^\#$ *exists, is there a unique* $\mathbb{P}$*-generic over* L *for some* $\mathbb{P} \in L$*?*

References

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  In preparation.

Thank you!