Forcing over Cohen's symmetric model

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Outline

- Cohen's first model is obtained by
  - Adding a sequence of Cohen reals, but
  - Forgetting their order
- We aim to
  - Study forcing over Cohen's first model
  - Understand forcing phenomena that come from Dedekind finite sets
- Joint work with Asaf Karagila (Norwich)

Cohen's first model

Let $L[G]$ be an $\mathrm{Add}(\omega, \omega)$-generic extension of L. Denote by

$$A = \{a_n \mid n \in \omega\},$$

where a_n is the nth Cohen real.

Cohen's first model is the submodel $M = L(A)$ of $L[G]$.

Recall that an infinite set B is called Dedekind-finite if there is no injection $f\colon \omega \to B$.

Fact

Every set in M is definable in $L[G]$ from A and finitely many a_n.

Fact

A is Dedekind-finite in M and in particular, A is not well-ordered in M.

Symmetric extensions

Definition

Let $\mathcal{G}$ be a group of automorphisms of $\mathbb{P}$. We say that $\mathcal{F}$ is a filter of subgroups over $\mathcal{G}$ if it is a filter on the lattice of subgroups. (It is closed under finite intersections and supergroups, and nonempty.)

We say that $\mathcal{F}$ is normal if whenever $H \in \mathcal{F}$ and $\pi \in \mathcal{G}$, then $\pi H \pi^{-1} \in \mathcal{F}$.

Definition

We call $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a symmetric system if $\mathbb{P}$ is a notion of forcing, $\mathcal{G}$ is a subgroup of $\mathrm{Aut}(\mathbb{P})$, and $\mathcal{F}$ is a normal filter of subgroups over $\mathcal{G}$.

Symmetric extensions

We denote by $\mathrm{stab}_{\mathcal{G}}(\dot{x})$ the group $\{\pi \in \mathcal{G} \mid \pi\dot{x} = \dot{x}\}$, the stabiliser of $\dot{x}$. We say that $\dot{x}$ is $\mathcal{F}$-symmetric if $\mathrm{stab}_{\mathcal{G}}(\dot{x}) \in \mathcal{F}$.

$\mathsf{HS}_{\mathcal{F}}$ denotes the class of hereditarily $\mathcal{F}$-symmetric names.

Definition

If G is $\mathbb{P}$-generic over V, then $\mathsf{HS}_{\mathcal{F}}^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}_{\mathcal{F}}\}$ is the symmetric extension induced by G.

$\mathsf{HS}_{\mathcal{F}}^G$ is a model of ZF and $\Vdash^{\mathsf{HS}}$ satisfies a version of the forcing theorem.

Lemma (The Symmetry Lemma)

Let $p \in \mathbb{P}$ be a condition, $\pi \in \mathrm{Aut}(\mathbb{P})$ and $\dot{x}$ a $\mathbb{P}$-name. Then

$$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi\dot{x}).$$

Cohen's first model

Our group of automorphisms is the group $\mathcal{G}$ of finitary permutations of ω acting on $\mathrm{Add}(\omega, \omega)$ in the natural way:

$$\pi p(\pi n, m) = p(n, m).$$

Take $\mathcal{F}$ to be the filter of subgroups generated by $\{\mathrm{fix}(s) \mid s \in [\omega]^{<\omega}\}$, where $\mathrm{fix}(s) = \{\pi \in \mathcal{G} \mid \pi{\restriction}s = \mathrm{id}\}$.

For each n, let $\dot{a}_n = \{\langle \check{m}, p\rangle \mid p(n, m) = 1\}$ be the canonical name for the nth Cohen real and $\dot{A} = \{(\dot{a}_n, 1) \mid n < \omega\}$ the canonical name for A.

Fact

$\pi \dot{a}_n = \dot{a}_{\pi n}$, and therefore $\pi \dot{A} = \dot{A}$, for all $\pi \in \mathcal{G}$.

Collapsing cardinals without collapsing cardinals

Collapsing

Definition

For two sets X and Y, let $\mathrm{Col}(X, Y)$ denote the set of partial functions $p\colon X \to Y$ such that $|p|$ is well-ordered and $|p| < |X|$, ordered by reverse inclusion.

For $X = A$ in Cohen's first model, the conditions are finite.

Theorem (Karagila, S.)

Let κ be an infinite cardinal and G a $\mathrm{Col}(A, \kappa)$-generic filter over M. Then M and $M[G]$ have the same sets of ordinals.

Proof sketch, part 1.

One can work with canonical conditions $\langle p, \dot{q}_f \rangle$, where q_f is induced by $f\colon \omega \to \kappa$.

Let $\langle p, \dot{q}_f \rangle$ be a condition which forces that $\dot{X}$ is a name for a set of ordinals. Let $s \in [\omega]^{<\omega}$ with $\mathrm{fix}(s) \subseteq \mathrm{stab}(\dot{X})$. We can assume that $s = \mathrm{supp}(p) = \mathrm{dom}(f)$. □

Collapsing

Proof sketch, part 2.

Suppose that $\langle p_0, \dot{q}_{f_0} \rangle$ and $\langle p_1, \dot{q}_{f_1} \rangle$ are two extensions of $\langle p, \dot{q}_f \rangle$. Again, we may assume that $\mathrm{supp}(p_i) = \mathrm{dom} f_i$ for $i < 2$.

If $p_1 \upharpoonright s = p_2 \upharpoonright s$, then the two must agree on any statement of the form $\check{\alpha} \in \dot{X}$. This is because there is an automorphism in $\mathrm{fix}(s)$ moving $\mathrm{supp}(p_0) \setminus s$ to be disjoint of $\mathrm{supp}(p_1)$. So $\langle \pi p_0, \pi \dot{q}_{f_0} \rangle$ is compatible with $\langle p_1, \dot{q}_{f_1} \rangle$ while $\pi \check{\alpha} = \check{\alpha}$ and $\pi \dot{X} = \dot{X}$. (This uses $\mathrm{dom}(f) = s$.)

In particular, if $\langle p', \dot{q}_{f'} \rangle \leq \langle p, \dot{q}_f \rangle$ and $\langle p', \dot{q}_{f'} \rangle \Vdash \check{\alpha} \in \dot{X}$, then $\langle p' \upharpoonright s, \dot{q}_{f' \upharpoonright s} \rangle = \langle p' \upharpoonright s, \dot{q}_f \rangle$ already forced this statement. The same holds for $\check{\alpha} \notin \dot{X}$.

Hence $\dot{X}^G$ is determined by finitely many Cohen reals a_n, thus $\dot{X}^G \in M$. □

Collapsing

Corollary

A is still Dedekind-finite after forcing with $\mathrm{Col}(A, \kappa)$.

Proof.

Suppose not, then there is an injective function $f\colon \omega \to A$ in $M[G]$, where G is $\mathrm{Col}(A, \kappa)$ over M. This function f can be coded as a real.

Since no new reals are added by the previous theorem, $f \in M$. Contradiction. □

How can one characterize preservation of Dedekind-finiteness of A for $\mathrm{Col}(A, \kappa)$ in general? And what about $\mathrm{Add}(A, 1) = \mathrm{Col}(A, 2)$?

Preserving Dedekind-finiteness

Preserving Dedekind-finiteness

Theorem (Karagila-S.)

Assuming ZF*, the following are equivalent for any Dedekind-finite set A.*

1. $[A]^{<\omega}$ *is Dedekind-finite.*
2. $\mathrm{Add}(A,1)$ *contains no infinite antichains.*
3. $\mathrm{Add}(A,1)$ *contains no countably infinite antichains.*
4. $\mathrm{Add}(A,1)$ *has the finite decision property.*
5. A *remains Dedekind-finite in any generic extension by* $\mathrm{Add}(A,1)$.
6. A *is not collapsed in any generic extension by* $\mathrm{Add}(A,1)$.
7. $\mathrm{Add}(A,1)$ *fails to add a Cohen real.*
8. $\mathrm{Add}(A,1)$ *fails to add a real.*
9. $\mathrm{Add}(A,1)$ *fails to add a set of ordinals.*
10. 2^A *is extremally disconnected.*

Preserving Dedekind-finiteness

This theorem admits an easy corollary, which is applicable to Cohen's first model.

Corollary

If A is a Dedekind-finite which can be linearly ordered, for instance a set of real numbers, then all conditions of the previous theorem hold.

Preserving Dedekind-finiteness

Definition

We say that $\mathrm{Add}(A,1)$ has the finite decision property if for all formulas $\varphi(\dot{x})$, the set $M^{\varphi(\dot{x})}$ of minimal elements of $N^{\varphi(\dot{x})} = \{p \mid p \Vdash \varphi(\dot{x})\}$ with respect to restriction is finite.

Lemma

The following are equivalent.

1. $[A]^{<\omega}$ *is Dedekind-finite.*
2. $\mathrm{Add}(A,1)$ *has the finite decision property.*

The proof uses the sunflower lemma, a finite version of the Δ-system lemma:

Lemma (Erdős–Rado 1960)

If a and b are positive integers, then any collection of $b!ab+1$ sets of size $\leq b$ contains a sunflower of size $>a$.

Preserving Dedekind-finiteness

Lemma

Let A be a Dedekind-finite set. The following conditions are equivalent.

1. $[A]^{<\omega}$ *is Dedekind-infinite.*
2. $\mathrm{Add}(A,1)$ *adds a set of ordinals.*

Proof.

$1 \implies 2$: Let $\vec{A} = \langle A_n \mid n < \omega \rangle$ be a disjoint sequence witnessing that $[A]^{<\omega}$ is Dedekind-infinite. Let

$$\dot{x} = \{\langle p, \check{n} \rangle \mid p[A_n] = \{0\}\}.$$

It is easy to see that $\dot{x}$ is a name for a Cohen real.

$2 \implies 1$: Suppose that 1 forces that $\dot{X}$ is a new subset of some ordinal η, and let $\varphi(\check{\alpha})$ denote the formula $\check{\alpha} \in \dot{X}$.

By the finite decision property, $M^{\varphi(\check{\alpha})}$ is finite for all $\alpha < \eta$.

However, the union of domains of conditions in $\bigcup_{\alpha<\eta} M^{\varphi(\check{\alpha})}$ is infinite, since $\dot{X}$ is a name for a new set of ordinals.

Thus it is easy to construct a disjoint sequence of finite subsets of A. □

Some open problems

Suppose that A is Dedekind-finite.

- Are there similar characterizations as above of the statement: $\mathrm{Col}(A, \kappa)$ preserves Dedekind-finiteness of A?

- If 2^A is compact, then $[A]^{<\omega}$ is Dedekind-finite. Is there a combinatorial characterization of compactness of 2^A?

References

- Asaf Karagila, Philipp Schlicht:
  How to have more things by forgetting how to count them
  submitted, available on arxiv.org