ITERATION OF L-PROPER FORCINGS

Any forcing $\mathbb{P}$ stands for one definable from a real parameter and $\mathbb{P}^M$ denotes the forcing in any set M containing this real. We call a forcing $\mathbb{P}$ L-proper if it is proper with respect to all models $L[x]$ containing the parameter, where x is a countable subsets of ω_1.[1]

In the following, we assume that each iterand is a forcing on the reals, i.e. its domain is a subset of $\mathcal{P}(\omega)$.[2] Moreover, we assume that each iterand is defined from a real in the ground model

Let $p \sqsubseteq q$ denote that p is an initial segment of q.

We call a $\mathbb{P}$-name σ *almost nice* if it is of the form $\sigma = \{(\check{n}, p) \mid p \in S_n\}$ for subsets S_n of $\mathbb{P}$. If $\mathbb{P}$ is an iterated forcing $\mathbb{P}_\alpha$, we inductively define σ to be *hereditarily almost nice* if each $p \in S_n$ is hereditarily almost nice. Since the set of hereditarily nice conditions is dense in $\mathbb{P}_\alpha$, we will from now write $\mathbb{P}_\alpha$ for this set. We will assume every name for a real is almost nice.

Suppose that $\pi\colon \alpha \to S$ is an order isomorphism. We define $\pi_*\colon \mathbb{P}_\alpha \to V$ by adding blocks of 1s in the gaps between S in the hereditary support of a condition $p \in \mathbb{P}_\alpha$. More precisely, we define $\pi(p)$ for $p \in \mathbb{P}_\alpha$ and $\pi(\sigma)$ for almost nice $\mathbb{P}_\alpha$-names σ by induction on α:

$$\pi_*(p)(j) = \begin{cases} \pi_*(p(i)) & \text{if } \pi(i) = j \\ 1 & \text{if } j \notin \operatorname{ran}(\pi) \end{cases}$$

$$\pi_*(\sigma) = \{(\check{n}, \pi_*(p)) \mid (\check{n}, p) \in \sigma\}).$$

We will also use π_* to denote $(\pi{\restriction}\bar{\alpha})_*$ for $\bar{\alpha} < \alpha$.

If $\bar{M}$ is transitive and $\pi\colon \bar{M} \to M$ is an isomorphism with $\alpha = \mathrm{Ord}^{\bar{M}}$ and $S = \mathrm{Ord}^M \subseteq \mathrm{Ord}$, then $(\pi{\restriction}\alpha)_* = \pi{\restriction}\mathbb{P}_\alpha^{\bar{M}}$ by the definition of $(\pi{\restriction}\alpha)_*$ via a trivial induction and elementarity of π.

Suppose that $\mathbb{P}_\gamma$ is a countable support iteration of L-proper forcings on the reals. Suppose that S is a countable subset of γ and $\pi\colon \alpha_S \to S$ is its uncollapse. We will also write $\pi(\alpha_S) = \sup(S)$ to simplify the notation below.

One can assume that the domain of each iterand is $\mathcal{P}(\omega)$. In case you don't want to make this assumption, we will work with the upwards closure $\hat{G}$ for $\leq$ of a filter G on $\mathbb{P}_\gamma$. Note that for $(p, \dot{q}) \in G$, $(1, \dot{q})$ might not be in G but is in $\hat{G}$.

Let $M = L[x]$ for a countable set x of ordinals such that $L[x]$ contains the parameter of the iterands of $\mathbb{P}_\alpha$.

The following defines $(M, \mathbb{P}_\alpha^M)$-generic conditions in $\mathbb{P}_\beta$ by translating the generic filter via π_*^{-1}. For $\beta \leq \gamma$ write $\beta^* = \sup(\pi^{-1}[\beta])$.

Definition 0.1. Suppose that M is a transitive model of ZFC^- with $\alpha_S \in M$ and $\beta \leq \gamma$. Call a condition $q \in \mathbb{P}_\beta$ *generic* if for every $\mathbb{P}_\beta$-generic filter G over V with $q \in G$, $\pi_*^{-1}[\hat{G}] \cap \mathbb{P}_{\beta^*}^M$ is a $\mathbb{P}_{\beta^*}^M$-generic filter over M.

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[1] This implies $\mathbb{P}^{L[x]} \subseteq \mathbb{P}$.

[2] This is no restriction. If you prefer to work with the standard definition of a tree forcing, take $\mathcal{P}(\omega^{<\omega})$ instead.

We fix some notation: For $p \in \mathbb{P}_\gamma$ and $\beta < \gamma$, let p^β denote $p{\restriction}[\beta, \gamma)$. We can identify p^β canonically with a $\mathbb{P}_\beta$-name such that $p{\restriction}\beta \Vdash_{\mathbb{P}_\beta} p^\beta \in \mathbb{P}_{[\beta,\gamma)}$. ($\mathbb{P}_{[\beta,\gamma)}$ means an iteration whose indices start with β.) Furthermore, if A is a subset of $\mathbb{P}_\gamma$ and $\{q{\restriction}\beta \mid q \in A\}$ is a maximal antichain in $\mathbb{P}_\beta$, then $p_A = \{(q^\beta, q{\restriction}\beta) \mid q \in A\}$ is a $\mathbb{P}_\beta$-name for an element of $\mathbb{P}_{[\beta,\gamma)}$. Let $N_{\beta,\gamma}$ denote the set of such names.

Let Γ be a $\mathbb{P}_\gamma$-name for the $\mathbb{P}_\gamma$-generic filter, and Γ^* a $\mathbb{P}_\gamma$-name for $\pi_*^{-1}[\hat{\Gamma}]$.

Lemma 0.2. *Suppose that $\pi(\alpha) \geq \beta$, $q \in \mathbb{P}_\beta$ is generic and $p \in N^M_{\beta^*,\alpha}$.*[3] *Then there is a generic $r \sqsupseteq q$ in $\mathbb{P}_{\pi(\alpha)}$ with*

$$q \Vdash_{\mathbb{P}_\beta} (r^\beta)^{\Gamma{\restriction}\beta} \leq \pi_*(p^{\Gamma^*{\restriction}\beta^*}).$$

Proof. The proof is a variant of preservation of properness. We proceed by induction on α. Since the claim is vacuous for $\alpha = 0$ or $\beta = \gamma$, we can assume that $0 < \alpha \leq \alpha_S$ and $\beta < \gamma$.

Case. $\alpha = \bar{\alpha} + 1$.

We will assume that $\pi(\bar{\alpha}) \geq \beta$, since otherwise the extension of q to $r \in \mathbb{P}_{\pi(\alpha)}$ by 1 is as required.

By the inductive hypothesis for $\bar{\alpha}$, we have a generic $\bar{r} \in \mathbb{P}_{\pi(\bar{\alpha})}$ with $\bar{r}{\restriction}\beta = q$ and

$$q \Vdash (\bar{r}^\beta)^{\Gamma{\restriction}\beta} \leq \pi_*(p^{\Gamma^*{\restriction}\beta^*}).$$

Let G be a $\mathbb{P}_{\pi(\bar{\alpha})}$-generic filter over V with $\bar{r} \in G$ and work in $V[G]$. Since $r = \bar{r}$ is generic, $G^* := \pi_*^{-1}[\hat{G}] \cap \mathbb{P}^M_{\bar{\alpha}}$ is a $\mathbb{P}^M_{\bar{\alpha}}$-generic filter over M. Since $\dot{\mathbb{P}}^G_{\pi(\bar{\alpha})}$ is L-proper in $V[G]$, there is an $(M[G^*], \mathbb{P}^{M[G^*]})$-generic condition $s \leq p(\bar{\alpha})^{G^*}$.

Back in V, pick a $\mathbb{P}_{\pi(\bar{\alpha})}$-name $\dot{s}$ such that $\bar{r}$ forces the above property of s. Then extend $\bar{r}$ to $r \in \mathbb{P}_{\pi(\alpha)}$ by letting $r(\pi(\bar{\alpha})) = \dot{s}$ and extending by 1 above $\pi(\bar{\alpha})$.

This ensures that r is generic. In more detail, suppose that G is $\mathbb{P}_{\pi(\bar{\alpha})}$-generic and $I = G{*}H$ is $\mathbb{P}_{\pi(\alpha)}$-generic over V with $r \in I$. Since $\bar{r} \in G$, $G^* := \pi_*^{-1}[\hat{G}]$ is a $\mathbb{P}^M_{\bar{\alpha}}$-generic filter over M. Let $\dot{t}$ denote $\mathbb{P}_{\bar{\alpha}}$-names for elements of the forcing at $\bar{\alpha}$, then $\pi_*(\dot{t})^G = \dot{t}^{G^*}$ by the definition of π_*. Since $r \in G{*}H$, we have $\dot{s}^G \in H$ and hence $H = \{\dot{t}^{G^*} \mid \pi_*(\dot{t})^G \in H\}$ is $\mathbb{P}_{\bar{\alpha}}$-generic over $M[G^*]$. So $\pi_*^{-1}[\hat{I}] = \{(p, \dot{t}) \in \mathbb{P}_\alpha \mid \pi_*(p) \in \hat{G},\ \pi_*(\dot{t})^G \in H\}$ is $\mathbb{P}_\alpha$-generic over M.[4]

Moreover, $\dot{s}^G \leq p(\pi(\bar{\alpha}))$ by the definition of $\dot{s}$, as required.

Case. α is a limit.

We can assume that $\sup(\pi[\alpha]) > \beta$, since otherwise the extension of q to $r \in \mathbb{P}_{\pi(\alpha)}$ by 1 is as required. Let n_0 be least with $\pi(\alpha_{n_0}) > \beta$.

Genericity at limits works just like in the usual proof of preservation of properness. We still give details. Suppose that $\alpha = \sup_{n\in\omega} \alpha_n$, where $\vec{\alpha} = \langle \alpha_n \mid n \in \omega \rangle$ is strictly increasing. Let $\vec{D} = \langle D_n \mid n \in \omega \rangle$ enumerate all dense open subsets $D \in M$ of $\mathbb{P}^M_\alpha$. Assume each set appears infinitely often. For each $n \in \omega$, there is some $p_{A_n} \in N^M_{\alpha_n,\alpha}$ with $A_n \in M$, $A_n \subseteq D_n$, by density of D_n. Moreover, we can choose A_{n+1} to refine A_n in the sense that for all $p \in A_n$, there is a subset S of $A_{n+1} \cap \{s \in \mathbb{P}^M_\alpha \mid s \leq p\}$ such that $\{s{\restriction}\alpha_{n+1} \mid p \in A_{n+1}\}$ is a maximal antichain below $p{\restriction}\alpha_{n+1}$.

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[3] This explicit notation is not strictly necessary; one could keep more closely to the usual proof of preservation of properness.

[4] Upwards closure of $\hat{I}$ is used since $(\pi_*(p), \pi_*(\dot{t}))$ might not be a condition.

Let $r_{n_0} = q$. We construct a $\sqsubseteq$-increasing sequence $\vec{r} = \langle r_n \mid n \geq n_0 \rangle$ by applying the induction hypothesis successively to r_n, α_n and p_{A_n} for $n \geq n_0$.

Let $G^* = \pi_*^{-1}[\hat{G}] \cap \mathbb{P}^M_\alpha$. Note that G^* is upwards closed by the upwards closure of $G^* {\restriction} \alpha_n$ given by the inductive hypothesis.

Claim. G^* is $\mathbb{P}^M_\alpha$-generic over M.

Proof. Since $r_n \in G {\restriction} \pi(\alpha_n)$, $G^* {\restriction} \alpha_n$ contains $p_n {\restriction} \alpha_n$ for a unique $p_n \in A_n$, for each $n \in \omega$. Since A_n refines A_m as above for $m < n$, we have $p_n \leq p_m$ for all $m \leq n$ and hence $p_m {\restriction} \alpha_n \in G^* {\restriction} \alpha_n$. Thus $p_n \in G^* \cap D_n$ for all $n \in \omega$, as required. □

Claim. G^* is a filter.

Proof. Let $p \in \mathbb{P}$. We claim that the set D_p of all $r \in \mathbb{P}_\alpha$ with (a) $r \leq p$ or (b) $\exists n \in \omega\ (r {\restriction} \alpha_n) \perp (p {\restriction} \alpha_n)$ is open dense. Towards a contradiction, suppose there is some $t \in \mathbb{P}_\alpha$ with $s \notin D_p$ for all $s \leq t$. Since (a) fails for all $r \leq t$, we have $t \perp p$. Since (b) fails for t, there is some $s_0 \leq (t {\restriction} \alpha_0), (p {\restriction} \alpha_0)$. Since $t \perp p$, s_0 forces that the tails of t and p are incompatible. Repeat this step above s_0 to obtain a $\sqsubseteq$-increasing sequence $\langle s_n \mid n \in \omega \rangle$ with union $s \leq t, p$. This contradicts $t \perp p$.

Now take any $p, q \in G^*$. Since D_p and D_q are dense, there is some $r \in G^* \cap D_p \cap D_q$. Since (b) fails for all $r \in G^*$, we have $r \leq p, q$, as required. □

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