Uncountable models and generalized Baire spaces

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The Ehrenfeucht-Fraïssé game

- We have two models $\mathfrak{A}$ and $\mathfrak{B}$, $A \cap B = \emptyset$.
- $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$ is the game:

I	x_0		x_1		$\cdots$	x_n		$\cdots$
II		y_0		y_1	$\cdots$		y_n	$\cdots$

- Rules:
  1. There are ω moves.
  2. $x_i, y_i \in A \cup B$.
  3. $x_i \in A$ iff $y_i \in B$.
  4. Player I wins if for some n the relation $x_i \leftrightarrow y_i$, $i < n$, does not extend to a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Otherwise II wins.

Partial (a.k.a. potential) isomorphism

TFAE:

1. $II \uparrow \mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$ i.e. II has a winning strategy in $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$.
2. $\mathfrak{A} \cong_p \mathfrak{B}$ i.e. $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic after some forcing.

(Ehrenfeucht 1957, Fraïssé 1953, Karp 1965)

Partial (a.k.a. potential) isomorphism

- If $\mathfrak{A}$ and $\mathfrak{B}$ are countable, then

$$\mathfrak{A} \cong \mathfrak{B} \iff \mathit{II} \uparrow \mathsf{EF}_\omega(\mathfrak{A}, \mathfrak{B}).$$

More generally, looking ahead

$\mathsf{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$ is defined similarly but there are κ moves.

More generally, looking ahead

TFAE:

1. $II \uparrow \mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$.
2. $\mathfrak{A} \cong_p^\kappa \mathfrak{B}$ i.e. $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic after some $< \kappa$-closed forcing.

More generally, looking ahead

- If $\mathfrak{A}$ and $\mathfrak{B}$ are of size $\leq \kappa$, then

$$\mathfrak{A} \cong \mathfrak{B} \iff II \uparrow \mathsf{EF}_\kappa(\mathfrak{A}, \mathfrak{B}).$$

Back to countable models

In an attempt to understand isomorphism of countable models (an analytic relation) better, we add an ordinal “clock” to the game EF_ω.

Adding a clock β

- $\mathrm{EF}_\omega^\beta(\mathfrak{A}, \mathfrak{B})$ is the following game:

I	x_0, α_0		x_1, α_1		$\cdots$	$x_n, \alpha_n = 0$		
II		y_0		y_1	$\cdots$			y_n

- Rules:
  1. There are only finitely many moves.
  2. $\beta > \alpha_0 > \ldots > \alpha_n = 0$.
  3. $x_i, y_i \in A \cup B$.
  4. $x_i \in A$ iff $y_i \in B$.
  5. Player I wins if for some m he played α_m and the relation $x_i \leftrightarrow y_i$, $i \le m$, does not extend to a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Otherwise II wins.

TFAE:

1. $II \uparrow \mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$.
2. $II \uparrow \mathrm{EF}_\omega^\beta(\mathfrak{A}, \mathfrak{B})$ for all $\beta < (|A| + |B|)^+$.

Scott Watershed

TFAE:

1. $\mathfrak{A} \not\cong_p \mathfrak{B}$.
2. There is β $(= \beta(\mathfrak{A}, \mathfrak{B}))$ such that *II* $\uparrow \mathrm{EF}_\omega^\beta(\mathfrak{A}, \mathfrak{B})$ but *I* $\uparrow \mathrm{EF}_\omega^{\beta+1}(\mathfrak{A}, \mathfrak{B})$.

Scott Watershed

- β is the Scott Watershed where advantage in the game $\mathrm{EF}_\omega^\gamma(\mathfrak{A}, \mathfrak{B})$ moves from II to I, more exactly
  - *II* $\uparrow \mathrm{EF}_\omega^\gamma(\mathfrak{A}, \mathfrak{B})$ for all $\gamma \leq \beta$.
  - *I* $\uparrow \mathrm{EF}_\omega^\gamma(\mathfrak{A}, \mathfrak{B})$ for all $\gamma > \beta$.

The "complexity" of a model: Scott Height

Fix $\mathfrak{A}$ (of any size). Let

$$\begin{aligned} \mathrm{SH}(\mathfrak{A}) &= sup\{\beta((\mathfrak{A}, \vec{a}), (\mathfrak{A}, \vec{b})) : \vec{a}, \vec{b} \in A^{<\omega}, (\mathfrak{A}, \vec{a}) \not\cong_p (\mathfrak{A}, \vec{b})\} \\ &= \text{the Scott Height of } \mathfrak{A}. \end{aligned}$$

[Scott, 1965]

Capturing isomorphism on countable models

For countable $\mathfrak{A}$ and $\mathfrak{B}$:

$$\mathfrak{A} \cong \mathfrak{B} \iff II \uparrow \mathsf{EF}_{\omega}^{\mathsf{SH}(\mathfrak{A})+\omega}(\mathfrak{A}, \mathfrak{B})$$

$$\iff \mathfrak{B} \models \psi_{\mathfrak{A}},$$

where $\psi_{\mathfrak{A}} \in L_{\omega_1 \omega}$ is the Scott Sentence of $\mathfrak{A}$.

[Scott, 1965]

Descriptive set theory of countable models

- An invariant subset of ω^ω is Borel iff it is definable in $L_{\omega_1\omega}$ [Scott, 1964].
- The orbit of a countable model is always Borel [Scott, 1964].
- There is a rich study of Borel and analytic (such as $\cong$) equivalence relations, starting from the above results of Scott.
- Countable ordinals play a central role.

A motivating question

- **Question:** Can the Scott analysis of countable models in terms of countable ordinals and sentences of $L_{\omega_1\omega}$ be extended from countable to uncountable?

Potential problems

- Need transfinite EF-games, i.e. EF_α, $\alpha > \omega$. But such games may be non-determined [Hyttinen and Väänänen, 1990], [Mekler et al., 1993], [Hyttinen et al., 2002].
- Clocks may have to be more general than just ordinals: we will use trees as clocks. But that leads to the question what is the order of trees like. Maybe it is not as nice as the order of ordinals?
- There is no maximal countable ordinal. Could there be a maximal tree of cardinality $\aleph_1$ without uncountable branches?

Ordering of trees

Suppose T and T' are trees i.e. partial orders in which the predecessors of every element are well-ordered and there is a unique root.

Definition

$T \leq T'$ if there is $\pi : T \to T'$ such that for all $t, u \in T$:

$$t <_T u \to \pi(t) <_{T'} \pi(u).$$

This π is called a weak embedding. If it is one-one it is called a strong embedding. Trees T and T' are equivalent, $T \equiv T'$, if $T \leq T'$ and $T' \leq T$.

- Ordinals i.e. well-founded trees (mod $\equiv$) form a proper class that is well-ordered by $\leq$.
- Trees of height κ without branches of length κ are the "ordinals" of GBS.
- There are $\leq$-incomparable trees of cardinality $\aleph_1$ without uncountable branches. [Todorčević and Väänänen, 1999]

Interesting classes of trees without κ-branches

A tree of height κ in which there is no branch of length κ is called:

1. A κ-Aronszajn tree if all levels are of size $< \kappa$.
2. A wide κ-Aronszajn tree if all levels are of size $\leq \kappa$.
3. A very wide κ-Aronszajn tree if all levels are of size $\leq \kappa^{<\kappa}$.

A kind of "successor" operation on trees

- $\sigma(T)$ denotes the tree of ascending chains in T, ordered by end-extension.
- $T < \sigma(T)$. (Kurepa)
- In many ways $\sigma(T)$ acts as the "successor" of the tree T.
- If T is a (wide) κ-Aronszajn tree, then $\sigma(T)$ is a very wide κ-Aronszajn tree.

We are ready to assign a clock-tree to the transfinite EF-game EF_κ.

EF-game with a clock-tree T

- $EF_T(\mathfrak{A}, \mathfrak{B})$ is the following game:

I	x_0, t_0		x_1, t_1		$\cdots$	x_ξ, t_ξ		$\cdots$
II		y_0		y_1	$\cdots$		y_ξ	$\cdots$

- Rules:
  1. There are as many moves as Player I can play.
  2. $t_0 < t_1 < \ldots < t_\xi < \ldots$ is an increasing chain in T.
  3. $x_i \in A$ iff $y_i \in B$.
  4. Player I wins if for some ξ he played t_ξ and the relation $x_i \leftrightarrow y_i$, $i \leq \xi$, does not extend to a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Otherwise II wins.
  5. If Player I cannot move (because he run out of branch in T), Player II wins.

TFAE:

1. $II \uparrow \mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$.
2. $II \uparrow \mathrm{EF}_T(\mathfrak{A}, \mathfrak{B})$ for all trees T without κ-branches, even assuming $|T| \leq 2^{(|A|+|B|)^{<\kappa}}$. [Hyttinen, 1987]

TFAE:

1. $I \uparrow \mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$.
2. $I \uparrow \mathrm{EF}_{\sigma(S)}(\mathfrak{A}, \mathfrak{B})$ for some S without κ-branches even with $|S| \leq (|A| + |B|)^{<\kappa}$. [Karttunen, 1984]

$T \leq T'$ syncs well with the EF-game:

Fix $\mathfrak{A}$ and $\mathfrak{B}$ (of any size).

- If $T \leq T'$ then

$$II \uparrow \mathrm{EF}_{T'}(\mathfrak{A}, \mathfrak{B}) \Rightarrow II \uparrow \mathrm{EF}_{T}(\mathfrak{A}, \mathfrak{B})$$

  and

$$I \uparrow \mathrm{EF}_{T}(\mathfrak{A}, \mathfrak{B}) \Rightarrow I \uparrow \mathrm{EF}_{T'}(\mathfrak{A}, \mathfrak{B}).$$

- If $I \uparrow \mathrm{EF}_{T'}(\mathfrak{A}, \mathfrak{B})$ and $II \uparrow \mathrm{EF}_{T}(\mathfrak{A}, \mathfrak{B})$, then $T < T'$.

[Hyttinen and Väänänen, 1990]

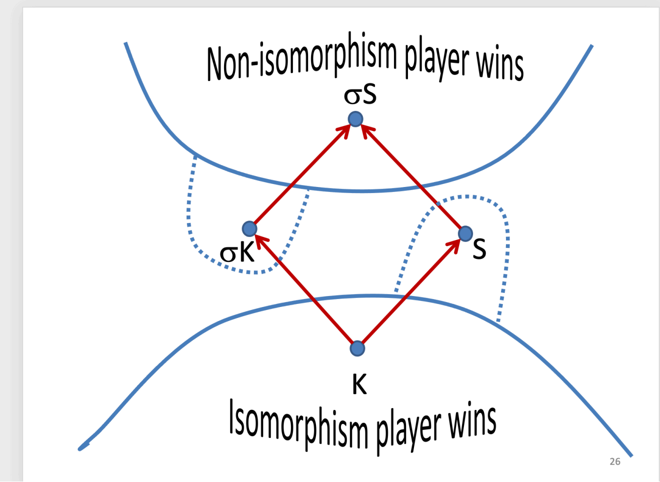
An analogue of the Scott Watershed

Theorem ([Hyttinen and Väänänen, 1990])

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are models of cardinality κ such that $\mathfrak{A} \not\cong \mathfrak{B}$. Then:

1. *There is a tree S such that $I \uparrow EF_{\sigma(S)}(\mathfrak{A}, \mathfrak{B})$ but $I \not\uparrow EF_S(\mathfrak{A}, \mathfrak{B})$. Moreover, $|S| \leq (|A| + |B|)^{<\kappa}$.*
2. *There is a tree $K \leq S$ such that $II \uparrow EF_K(\mathfrak{A}, \mathfrak{B})$ but $II \not\uparrow EF_{\sigma(K)}(\mathfrak{A}, \mathfrak{B})$. Moreover, $|K| \leq 2^{(|A|+|B|)^{<\kappa}}$.*

Given $\mathfrak{A}$ and $\mathfrak{B}$, the class of trees is divided into regions of advantage in the EF-game.



Two[1] analogues of Scott Height [Hyttinen and Väänänen, 1990]

Let $\mathfrak{A}$ be a model of cardinality κ.
A tree T without κ-branches is a:

- universal equivalence tree of $\mathfrak{A}$ if for all $\mathfrak{B}$ of cardinality κ:

$$\mathfrak{A} \cong \mathfrak{B} \iff II \uparrow \mathrm{EF}_T(\mathfrak{A}, \mathfrak{B}).$$

- universal non-equivalence tree of $\mathfrak{A}$ if for all $\mathfrak{B}$ of cardinality κ:

$$\mathfrak{A} \not\cong \mathfrak{B} \iff I \uparrow \mathrm{EF}_T(\mathfrak{A}, \mathfrak{B}).$$

[1] Because of non-determinacy we have two rather than one analogue.

Linear orders

Theorem ([Hyttinen and Tuuri, 1991])

Assume CH. There is a linear order of cardinality $\aleph_1$ *without a universal equivalence tree of cardinality* $\aleph_1$*.*

$\aleph_1$-separable[2] abelian group

Theorem ([Eklof et al., 1995])

1. *PFA* $\vdash$ *every* $\aleph_1$*-separable abelian group of cardinality* $\aleph_1$ *has a universal equivalence tree of cardinality* $\aleph_1$*.*
2. $\diamond \vdash$ *there is an* $\aleph_1$*-separable abelian group of cardinality* $\aleph_1$ *without a universal equivalence tree of cardinality* $\aleph_1$*.*

Such results put a bound on how to put invariants on such groups.

[2] Every countable subset is contained in a countable free direct summand.

The idea is that the bigger a universal equivalence tree is, the more complicated the model is. If there is no universal equivalence tree of cardinality κ (there is always one of cardinality $2^{\kappa^{<\kappa}}$), then the model is in a sense maximally complicated.

Open problem

- **Question:** Given a (wide) κ-Aronszajn tree T, are there models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ such that $II \uparrow \mathrm{EF}_T(\mathfrak{A}, \mathfrak{B})$ but $\mathfrak{A} \not\cong \mathfrak{B}$?

- Yes, if $\kappa = \omega$. [Karp, 1965]
- Yes, if $\kappa^{<\kappa} = \kappa$. [Hyttinen and Tuuri, 1991]
- Yes, if T is not too big, e.g. has height $< \kappa$. [Shelah, 2008]
- Open also if "wide" is dropped or replaced by "very wide".

Connection to Δ^1_1

Theorem ([Mekler and Väänänen, 1993])

Assume CH and $R \subseteq \omega_1 \times \omega_1$. The following conditions are equivalent:

1. *The model (ω_1, R) has a universal non-equivalence tree of cardinality $\aleph_1$.*
2. *The orbit of R in $\omega_1^{\omega_1}$ is $\mathbf{\Delta}^1_1$ in the Generalized Baire Space $\omega_1^{\omega_1}$.*

Canary trees

Let $\Phi(\omega_1)$ be the result of replacing in the order type of ω_1 every element by a copy of the rationals.

Theorem ([Mekler and Väänänen, 1993])

Assume CH. TFAE:

1. $\Phi(\omega_1)$ *has a universal equivalence tree of cardinality* $\aleph_1$.
2. *There is a wide Aronszajn tree (a "Canary" tree) which is* $\leq$*-above every tree of the form* $T(A)$, $A \subseteq \omega_1$ *co-stationary.*

Theorem

$\kappa > \omega$ regular. Γ a countable complete first order theory.

1. *If Γ is classifiable, then for every model $\mathfrak{A}$ of Γ of cardinality $\aleph_1$ there is a tree of cardinality $\aleph_1$ and of countable height which is a universal non-equivalence tree for $\mathfrak{A}$.* [Shelah, 2023]
2. *If $\kappa^{<\kappa} = \kappa$ and Γ is non-classifiable , then Γ has a model $\mathfrak{A}$ of cardinality κ such that no wide κ-Aronszajn tree is a universal equivalence tree for $\mathfrak{A}$.* [Hyttinen and Tuuri, 1991]
3. *It is consistent, relative to the consistency of ZF, that if Γ is unsuperstable, then Γ has a model $\mathfrak{A}$ of cardinality κ such that no wide κ-Aronszajn tree is a universal non-equivalence tree for $\mathfrak{A}$.* [Hyttinen and Tuuri, 1991]

- The question of existence of universal (non)equivalence trees for uncountable models emphasises the need to understand what kind of classes of trees have maximal (i.e. universal) trees.
- Problem: Given a class of trees, is there a maximal tree in the class under weak embeddings?

- MA_{ω_1} implies there is no maximal $\aleph_1$-Aronszajn tree. [Todorčević, 2007]
- MA_{ω_1} implies there is no maximal wide $\aleph_1$-Aronszajn tree. [Džamonja and Shelah, 2021]
- $\kappa^{<\kappa} = \kappa$ implies there is no maximal wide κ-tree. (Because $|\sigma(T)| \leq |T|^{<\kappa}$.)
- There is no maximal very wide κ-Aronszajn tree. (Because of σ.)
- Assume $V = L$ and κ regular but not weakly compact. No wide κ-Aronszajn tree T is maximal, for there is always a κ-Souslin tree S such that $S \not\leq T$. [Todorčević and Väänänen, 1999], [Ben-Neria - Magidor - Väänänen 2023].
- Is it consistent to have a maximal wide κ-Aronszajn tree?

Theorem (Ben-Neria - Magidor - Väänänen 2023)

Assuming the consistency of a weakly compact cardinal above a regular uncountable cardinal μ, it is consistent that there exists a maximal wide μ^+-Aronszajn tree, i.e. a tree of height and cardinality μ^+ with no branches of length μ^+, into which every wide μ^+-Aronszajn tree can be (strongly) embedded.

Universally Baire sets in Generalized Baire Spaces

Joint work with Menachem Magidor

- $\omega_1^{\omega_1}$ topology by initial segments.
- κ^κ topology by initial segments.
- Nowhere dense ... as usual.
- κ-meager ...as usual.
- Σ^1_1 ... as usual, but note that every subset of $\omega_1^{\omega_1}$ may be Σ^1_1 (Schindler).
- Strongly $\Sigma^1_1 = \Sigma_1$ over H_{κ^+}.

Recap: Universally Baire in ω^ω

- $A \subseteq \omega^\omega$ is universally Baire if $f^{-1}[A]$ is Baire in E for every continuous $f : E \to \omega^\omega$.
- Schilling-Vaught 1983, Feng-Magidor-Woodin 1992.
- A σ-algebra in the intersection of Lebesque measurable sets and Baire sets.
- Σ^1_1-sets are UB.
- Large cardinals imply projective sets are UB.

Recap: Forcing definition of UB

$A \subseteq \omega^\omega$ is universally Baire iff for every $\mathbb{P}$ there is a $\mathbb{P}$-term τ such that for all countable $M \prec H_\theta$, θ big, with $A, \mathbb{P} \in M$, and for all G, $\mathbb{P}$-generic over M, we have

$$[\tau]_G = A \cap M[G].$$

Generalization to κ^κ, κ regular

We say that $A \subseteq \kappa^\kappa$ is UB$(\mathcal{P}, \mathcal{M}, \mathcal{G})$, if for every $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-term τ such that for all $M \prec H_\theta$, θ big, such that $|M| = \kappa$, $M \in \mathcal{M}$ and $A, \kappa, \mathbb{P} \in M$, and for all $G \in \mathcal{G}$, $\mathbb{P}$-generic over M, we have

$$[\tau]_G = A \cap M[G].$$

$\mathrm{UB}(\mathcal{P}, \mathcal{M}, \mathcal{G})$

Typical cases:

- $\mathcal{P}$ is the class of κ-closed po-sets (CL_κ), or just stationary preserving (SP) po-sets.
- $\mathcal{M}$ is internally κ-closed models (IC_κ). $M = \bigcup_{\alpha<\kappa} M_\alpha$, $|M_\alpha| < \kappa$, $\{\langle M_\beta : \beta < \alpha\rangle\} \cup \{M_\alpha\} \subseteq M_{\alpha+1}$.
- $\mathcal{G}=$ all generics, $\mathcal{G}=$ stationary correct generics (SCO).

A is κ-universally Baire if it satisfies $\mathrm{UB}(\mathrm{CL}_\kappa, \mathrm{IC}_\kappa, \mathrm{all})$.

Properties:

- Baire property.
- Bernstein property.
- Topological characterization.
- $\mathrm{CLUB} = \{f \in \kappa^\kappa : f^{-1}(0) \text{ contains a club }\}$ is not here.
- The set of (codes of) wide Aronszajn trees is here, assuming Martin's Axiom and not-CH, but not assuming $\diamondsuit$.
- Same with the set of (codes of) Souslin trees.

Theorem (Bernstein Property)

Suppose κ is regular. If $A \subseteq \kappa^\kappa$ is κ-universally Baire, then either A or $\kappa^\kappa \setminus A$ contains a copy of 2^κ.

Need a weaker concept of universal Baireness in order that CLUB, a natural and central concept, would be included.

A is weakly κ-universally Baire if it satisfies $\mathrm{UB}(\mathrm{SP}, \mathrm{IC}_\kappa, \mathrm{SCO})$.

Properties:

- CLUB is here, so this does not imply Baire property.
- A weak Bernstein property in $\omega_1^{\omega_1}$, assuming MM^{++}.
- $V = L$ implies every Σ^1_1 set is here (for $\kappa = \lambda^+$, λ regular.)
- Can always force "SLN not here".

Theorem (A weak Bernstein property)

Assume MM^{++}. If $A \subseteq \omega_1^{\omega_1}$ is weakly universally Baire, then either A of $\omega_1^{\omega_1} \setminus A$ contains an ω_1-rake.

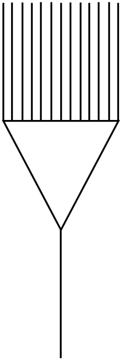


Figure: An ω_1-rake.

Conclusion: No large cardinals can imply that all strongly Σ^1_1-sets are weakly ω_1-universally Baire.

We need something weaker than "weak".

The 1st version

SP(MM) is the class of stationary preserving po-sets that force MM.

A is said to be very weakly κ-universally Baire if it satisfies UB(SP(MM), IC_κ, SCO).

Properties when $\kappa = \omega_1$:

- Every strongly Σ^1_1 set is here, assuming a proper class of Woodin cardinals.
- A weak Bernstein property, assuming MM^{++} and a supercompact.

The 2nd version

SP($\star$): stationary preserving po-sets forcing ($\star$).

Definition

A is very weakly κ-universally Baire if it satisfies $\mathrm{UB}(\mathrm{SP}(\star), \mathrm{IC}_\kappa, \mathrm{SCO})$.

Properties when $\kappa = \omega_1$:

- Every strongly projective set is here, assuming ($\star$).
- A weak Bernstein property, assuming MM^{++} and a supercompact.

Conclusions

- Move from countable to uncountable is full of troubles and surprises, as can be expected.
- By using trees as analogues of ordinals we can go around some problems.
- MM^{++} and $(\star)$ are natural frameworks to develop descriptive set theory in generalized Baire spaces, when CH is not assumed.

Thank you!

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