Covering with Closed Sets

Dorottya Sziráki
Joint work in progress with Philipp Schlicht

7^{th} Workshop on Generalized Baire Spaces
Bristol, 08 February 2024

Motivation

Very general dichotomies have emerged for Polish spaces which imply several old and new theorems in descriptive set theory.

General Aims:

- Versions of these dichotomies for generalized Baire spaces.
- Lift known applications to the uncountable setting.
- New applications.

The setting:

κ always denotes an infinite cardinal with $\kappa^{<\kappa} = \kappa$.

${}^{\kappa}d$ always has the bounded topology τ_b for any discrete topological space d, with basic open sets $N_t := \{x \in {}^{\kappa}d : t \subseteq x\}$, where $t \in {}^{<\kappa}d$.

The Closed-Sets Covering Property

$\mathcal{F}$ always denotes a family of closed subsets of ${}^{\kappa}\kappa$. $\mathcal{I}_{\mathcal{F}}$ is the κ-ideal generated by $\mathcal{F}$ (i.e., the closure of $\mathcal{F}$ under taking unions of size κ and subsets).

Definition. Suppose $X \subseteq {}^{\kappa}\kappa$, C is a class.

$\mathsf{CCP}^{\mathsf{C}}_{\kappa}(X)$: For any family $\mathcal{F}$ of closed subsets of ${}^{\kappa}\kappa$, either $X \in \mathcal{I}_{\mathcal{F}}$ or X has an $\mathcal{I}_{\mathcal{F}}$-positive subset $Y \in \mathsf{C}$.

The Closed-Sets Covering Property

$\mathcal{F}$ always denotes a family of closed subsets of ${}^\kappa\kappa$. $\mathcal{I}_\mathcal{F}$ is the κ-ideal generated by $\mathcal{F}$

(i.e., the closure of $\mathcal{F}$ under taking unions of size κ and subsets).

Definition. Suppose $X \subseteq {}^\kappa\kappa$, C is a class.

$\mathsf{CCP}_\kappa^\mathsf{C}(X)$: For any family $\mathcal{F}$ of closed subsets of ${}^\kappa\kappa$, either $X \in \mathcal{I}_\mathcal{F}$ or X has an $\mathcal{I}_\mathcal{F}$-positive subset $Y \in \mathsf{C}$.

Theorem (Louveau)

$\mathsf{CCP}_\omega^{\mathbf{\Sigma}^1_1}(X)$ *holds for all subsets of* ${}^\omega\omega$ *in Solovay's model.*

The Closed-Sets Covering Property

$\mathcal{F}$ always denotes a family of closed subsets of ${}^\kappa\kappa$. $\mathcal{I}_\mathcal{F}$ is the κ-ideal generated by $\mathcal{F}$ (i.e., the closure of $\mathcal{F}$ under taking unions of size κ and subsets).

Definition. Suppose $X \subseteq {}^\kappa\kappa$, C is a class.

$\mathsf{CCP}_\kappa^{\mathsf{C}}(X)$: For any family $\mathcal{F}$ of closed subsets of ${}^\kappa\kappa$, either $X \in \mathcal{I}_\mathcal{F}$ or X has an $\mathcal{I}_\mathcal{F}$-positive subset $Y \in \mathsf{C}$.

Theorem (Louveau)

$\mathsf{CCP}_\omega^{\mathbf{\Sigma}^1_1}(X)$ *holds for all subsets of* ${}^\omega\omega$ *in Solovay's model.*

Theorem (Solecki)

$\mathsf{CCP}_\omega^{\mathbf{\Pi}^0_2}(X)$ *holds for all analytic subsets of* ${}^\omega\omega$.

Hence $\mathsf{CCP}_\omega^{\mathbf{\Sigma}^1_1}(X) \iff \mathsf{CCP}_\omega^{\mathbf{\Pi}^0_2}(X)$ for all $X \subseteq {}^\omega\omega$.

The Closed-Sets Covering Property

Definition. Suppose $X \subseteq {}^{\kappa}\kappa$.

$\mathsf{CCP}_{\kappa}(X)$: For any family $\mathcal{F}$ of closed subsets of ${}^{\kappa}\kappa$, either $X \in \mathcal{I}_{\mathcal{F}}$ or there is a continuous function $f : {}^{\kappa}\kappa \to X$ with $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$.

The Closed-Sets Covering Property

Definition. Suppose $X \subseteq {}^{\kappa}\kappa$.

$\mathsf{CCP}_{\kappa}(X)$: For any family $\mathcal{F}$ of closed subsets of ${}^{\kappa}\kappa$, either $X \in \mathcal{I}_{\mathcal{F}}$ or there is a continuous function $f : {}^{\kappa}\kappa \to X$ with $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$.

$\mathsf{CCP}_{\kappa}(X, \mathsf{Def}_{\kappa})$ is the restriction to definable families $\mathcal{F}$ of closed sets.

By "definable", we always mean "definable from a κ-sequence of ordinals".

The Closed-Sets Covering Property

Definition. Suppose $X \subseteq {}^{\kappa}\kappa$.

$\mathsf{CCP}_{\kappa}(X)$: For any family $\mathcal{F}$ of closed subsets of ${}^{\kappa}\kappa$, either $X \in \mathcal{I}_{\mathcal{F}}$ or there is a continuous function $f : {}^{\kappa}\kappa \to X$ with $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$.

$\mathsf{CCP}_{\kappa}(X, \mathsf{Def}_{\kappa})$ is the restriction to definable families $\mathcal{F}$ of closed sets.

By "definable", we always mean "definable from a κ-sequence of ordinals".

Theorem (Shlicht, Sz)

After a Lévy-collapse of λ to κ^{+}, the following hold for all definable $X \subseteq {}^{\kappa}\kappa$:

- $\mathsf{CCP}_{\kappa}(X)$ *if λ is Mahlo.*
- $\mathsf{CCP}_{\kappa}(X, \mathsf{Def}_{\kappa})$ *if λ is inaccessible.*

It's all the same in the countable setting

Lemma

$\mathsf{CCP}_\omega(X) \iff \mathsf{CCP}_\omega^{\boldsymbol{\Sigma}^1_1}(X)$ *for all* $X \subseteq {}^\omega\omega$.

Proof.

It suffices to show $\mathsf{CCP}_\omega(\boldsymbol{\Sigma}^1_1)$. Let X be an $\mathcal{I}_\mathcal{F}$-positive analytic set, and let $f : {}^\omega\omega \to X$ be a continuous surjection. For all $t \in {}^{<\omega}\omega$, take an infinite maximal antichain A_t of nodes u in ${}^{<\omega}\omega$ with $t \subseteq u$ and $f(N_u) \in \mathcal{I}^+_\mathcal{F}$.

Construct a strict order preserving map $\phi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $\langle \phi(t^\frown\langle i\rangle) : i < \omega\rangle$ enumerates $A_{\phi(t)}$ without repetitions for each $t \in {}^{<\omega}\omega$.

$[\phi](x) := \bigcup_{t \subsetneq x} \phi(t)$ for all $x \in {}^\omega\omega$. Then $g := f \circ [\phi]$ is a continuous map from ${}^\omega\omega$ to X with $g(N_t) \in \mathcal{I}^+_\mathcal{F}$ for all $t \in {}^{<\omega}\omega$. □

It's all the same in the countable setting

Lemma

$\mathsf{CCP}_\omega(X) \iff \mathsf{CCP}_\omega^{\mathbf{\Sigma}_1^1}(X)$ *for all* $X \subseteq {}^\omega\omega$.

It's all the same in the countable setting

Lemma

$\mathsf{CCP}_\omega(X) \iff \mathsf{CCP}_\omega^{\boldsymbol{\Sigma}^1_1}(X)$ *for all* $X \subseteq {}^\omega\omega$.

Suppose $\kappa > \omega$. Let C_κ denote the class of continuous images of ${}^\kappa\kappa$.

Question

Does $\mathsf{CCP}_\kappa(X)$ follow from either $\mathsf{CCP}_\kappa^{\boldsymbol{\Sigma}^1_1}(X)$ or $\mathsf{CCP}_\kappa^{\mathsf{C}_\kappa}(X)$ for all X, κ?

It's all the same in the countable setting

Lemma

$\mathsf{CCP}_\omega(X) \iff \mathsf{CCP}_\omega^{\mathbf{\Sigma}^1_1}(X)$ *for all* $X \subseteq {}^\omega\omega$.

Suppose $\kappa > \omega$. Let C_κ denote the class of continuous images of ${}^\kappa\kappa$.

Question

Does $\mathsf{CCP}_\kappa(X)$ follow from either $\mathsf{CCP}_\kappa^{\mathbf{\Sigma}^1_1}(X)$ or $\mathsf{CCP}_\kappa^{\mathsf{C}_\kappa}(X)$ for all X, κ?

By Solecki's result, $\mathsf{CCP}_\omega(X) \Leftrightarrow \mathsf{CCP}_\omega^{\mathbf{\Pi}^0_2}(X)$.

Question

Does $\mathsf{CCP}_\kappa(X)$ imply $\mathsf{CCP}_\kappa^{\mathbf{\Pi}^0_2}(X)$ for all X, κ?

Example: the κ-perfect set property

Let $\mathsf{CCP}_\kappa(X, \mathcal{F})$ and $\mathsf{CCP}^{\mathsf{C}}_\kappa(X, \mathcal{F})$ denote the versions for a single family $\mathcal{F}$ of closed sets.

Example: the κ-perfect set property

Let $\mathsf{CCP}_\kappa(X,\mathcal{F})$ and $\mathsf{CCP}^{\mathsf{C}}_\kappa(X,\mathcal{F})$ denote the versions for a single family $\mathcal{F}$ of closed sets.

Suppose $\mathcal{F}$ is the family of singletons.

Then $\mathsf{CCP}_\kappa(X,\mathcal{F})$ is equivalent to the κ-perfect set property $\mathrm{PSP}_\kappa(X)$:
either $|X|\leq\kappa$ or X contains a κ-perfect subset (i.e., the body $[T]$ of a cofinally splitting $<\kappa$-closed tree T).

Example: the κ-perfect set property

Let $\mathsf{CCP}_\kappa(X,\mathcal{F})$ and $\mathsf{CCP}^{\mathsf{C}}_\kappa(X,\mathcal{F})$ denote the versions for a single family $\mathcal{F}$ of closed sets.

Suppose $\mathcal{F}$ is the family of singletons.

Then $\mathsf{CCP}_\kappa(X,\mathcal{F})$ is equivalent to the κ-perfect set property $\mathrm{PSP}_\kappa(X)$:
either $|X| \leq \kappa$ or X contains a κ-perfect subset (i.e., the body $[T]$ of a cofinally splitting $<\kappa$-closed tree T).

If $V = L$, then

- $\mathrm{PSP}_\kappa(\mathbf{\Sigma}^1_1)$ fails for all $\kappa = \kappa^{<\kappa} > \omega$ (Friedman, Hyttinen, Kulikov).
- $\mathrm{PSP}_\kappa(\mathsf{C}_\kappa)$ fails for $\kappa = \omega_2$ (Lücke, Schlicht).

So $\mathsf{CCP}_\kappa(X,\mathcal{F})$ is strictly weaker than both $\mathsf{CCP}^{\mathbf{\Sigma}^1_1}_\kappa(X,\mathcal{F})$ and $\mathsf{CCP}^{\mathsf{C}_\kappa}_\kappa(X,\mathcal{F})$.

Example: the asymmetric κ-Baire property

$X \subseteq {}^\kappa\kappa$ has the κ-Baire property if there is an open set $U \subseteq {}^\kappa\kappa$ such that $X \triangle U$ is κ-meager (i.e. the union of κ-many nowhere dense sets).

The κ-Baire property holds for κ-Borel sets, but it fails for κ-analytic sets when $\kappa > \omega$:

Example (Halko, Shelah)

$$\mathrm{Club}_\kappa := \{x \in {}^\kappa\kappa \ : \ \{\alpha < \kappa : x(\alpha) > 0\} \text{ contains a club}\}$$

does not have the κ-Baire property. In fact, it is not κ-meager or κ-comeager in any open set.

The Banach-Mazur game $G^{**}_\kappa(X)$ of length κ is:

I	t_0		t_2		$\dots$		$t_{2\alpha}$		$\dots$	
II		t_1		t_3		$\dots$		$t_{2\alpha+1}$		$\dots$

where $t_\alpha \in {}^{<\kappa}\kappa$. **I** wins if $\bigoplus_{\alpha<\kappa} t_\alpha := t_0{}^\frown t_1{}^\frown \dots t_\alpha{}^\frown \dots$ is in X.

Definition (Asymmetric κ-Baire property)

$\mathsf{ABP}_\kappa(X)$ states that $G^{**}_\kappa(X)$ is determined.

Example: the asymmetric κ-Baire property

A strict order preserving map $\phi : {}^{<\kappa}\kappa \to {}^{<\kappa}\kappa$ is dense if $\{\phi(t^\frown\langle i\rangle) : i < \kappa\}$ is dense above $\phi(t)$ for all $t \in {}^{<\kappa}\kappa$.

Lemma (Kovachev; Schlicht)

$\mathsf{ABP}_\kappa(X)$ *holds if and only if either*

- X *is κ-meager, or*
- *there exists a dense strict order preserving map $\phi : {}^{<\kappa}\kappa \to {}^{<\kappa}\kappa$ with $\mathrm{ran}([\phi]) \subseteq X$.*

Example: the asymmetric κ-Baire property

In more detail:

Lemma (Kovachev; Schlicht)

TFAE for any $X \subseteq {}^{\kappa}\kappa$:

- X *is κ-meager.*
- **II** *has a winning strategy in* $G^{**}_{\kappa}(X)$.
- *There exists a dense continuous strict order preserving map* $\phi : {}^{<\kappa}\kappa \to {}^{<\kappa}\kappa$ *with* $\phi(\emptyset) = \emptyset$ *and* $\operatorname{ran}([\phi]) \subseteq {}^{\kappa}\kappa \setminus X$.

Lemma (Schlicht)

TFAE for any $X \subseteq {}^{\kappa}\kappa$:

- **I** *has a winning strategy in* $G^{**}_{\kappa}(X)$.
- *There exists a dense strict order preserving map* $\phi : {}^{<\kappa}\kappa \to {}^{<\kappa}\kappa$ *with* $\operatorname{ran}([\phi]) \subseteq X$.

Example: the asymmetric κ-Baire property

Let $\mathcal{F}$ be the family of closed nowhere dense subsets of ${}^{\kappa}\kappa$. Then $\mathcal{I}_{\mathcal{F}}$ consists of all κ-meager sets.

Lemma

The existence of the following objects is equivalent for all $X \subseteq {}^{\kappa}\kappa$:

1. *a dense strict order preserving map $\phi : {}^{<\kappa}\kappa \to {}^{<\kappa}\kappa$ with $\mathrm{ran}([\phi]) \subseteq X$,*
2. *a continuous map $f : {}^{\kappa}\kappa \to X$ with $f(N_t) \in \mathcal{I}^{+}_{\mathcal{F}}$ for all $t \in {}^{<\kappa}\kappa$.*

Hence $\mathsf{ABP}_{\kappa}(X) \iff \mathsf{CCP}_{\kappa}(X, \mathcal{F})$.

Example: the asymmetric κ-Baire property

Let $\mathcal{F}$ be the family of closed nowhere dense subsets of ${}^{\kappa}\kappa$. Then $\mathcal{I}_{\mathcal{F}}$ consists of all κ-meager sets.

Lemma

The existence of the following objects is equivalent for all $X \subseteq {}^{\kappa}\kappa$:

1. *a dense strict order preserving map $\phi : {}^{<\kappa}\kappa \to {}^{<\kappa}\kappa$ with $\mathrm{ran}([\phi]) \subseteq X$,*
2. *a continuous map $f : {}^{\kappa}\kappa \to X$ with $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$.*

Hence $\mathsf{ABP}_{\kappa}(X) \iff \mathsf{CCP}_{\kappa}(X, \mathcal{F})$.

Proof sketch. $(1) \Rightarrow (2)$: $f := [\phi]$.

Example: the asymmetric κ-Baire property

$(2) \Rightarrow (1)$: Since $R := {}^{<\kappa}\kappa$ has size κ, it suffices to construct a dense strict order preserving map $\phi : {}^{<\kappa}R \to {}^{<\kappa}\kappa$ with $\mathrm{ran}([\phi]) \subseteq X$.

Construct strict order preserving maps $\phi : {}^{<\kappa}R \to {}^{<\kappa}\kappa$ and $\iota : {}^{<\kappa}R \to {}^{<\kappa}R$ such that ι is continuous (i.e., $\iota(t) = \bigcup_{s \subsetneq t} \iota(s)$ for all t of limit lengths) and

- $\phi(t^\frown\langle r\rangle) \supseteq \phi(t)^\frown r$,
- $f(N_{\iota(t^\frown\langle r\rangle)}) \subseteq N_{\phi(t)^\frown r}$

for all $t \in {}^{<\kappa}R$ and $r \in R$. Then ϕ will be a dense map with $[\phi] = f \circ [\iota]$.

Example: the asymmetric κ-Baire property

$(2) \Rightarrow (1)$: Since $R := {}^{<\kappa}\kappa$ has size κ, it suffices to construct a dense strict order preserving map $\phi : {}^{<\kappa}R \to {}^{<\kappa}\kappa$ with $\mathrm{ran}([\phi]) \subseteq X$.

Construct strict order preserving maps $\phi : {}^{<\kappa}R \to {}^{<\kappa}\kappa$ and $\iota : {}^{<\kappa}R \to {}^{<\kappa}R$ such that ι is continuous (i.e., $\iota(t) = \bigcup_{s \subsetneq t} \iota(s)$ for all t of limit lengths) and

- $\phi(t^\frown\langle r\rangle) \supseteq \phi(t)^\frown r$,
- $f(N_{\iota(t^\frown\langle r\rangle)}) \subseteq N_{\phi(t)^\frown r}$

for all $t \in {}^{<\kappa}R$ and $r \in R$. Then ϕ will be a dense map with $[\phi] = f \circ [\iota]$.

The construction:

If $\iota(t)$ and $\phi(t)$ have been constructed, let $u := \begin{cases} \phi(s)^\frown r & \text{if } t = s^\frown\langle r\rangle, \\ \bigcup_{s \subsetneq t} \phi(s) & \text{if } \mathrm{lh}(t) \in \mathrm{Lim}. \end{cases}$

Since $f(N_{\iota(t)}) \subseteq N_u$ is somewhere dense, take $\phi(t) \supseteq u$ such that $f(N_{\iota(t)})$ is dense in $N_{\phi(t)}$. Find $\iota(t^\frown\langle r\rangle)$ using that $f(N_{\iota(t)}) \cap N_{\phi(t)^\frown\langle r\rangle} \neq \emptyset$ and the continuity of f.

$\square$

Example (continued)

$\mathsf{ABP}_\kappa(\mathrm{Club}_\kappa)$ holds, but $\mathsf{CCP}_\kappa^{\mathsf{Borel}_\kappa}(\mathrm{Club}_\kappa, \mathcal{F})$ fails: If Club_κ contained an $\mathcal{I}_\mathcal{F}$-positive κ-Borel subset, then it would be κ-comeager in some open set since κ-Borel sets have the κ-Baire property.

This shows that $\mathsf{CCP}_\kappa(X, \mathcal{F}')$ does not always imply $\mathsf{CCP}_\kappa^{\mathbf{\Pi}^0_2}(X, \mathcal{F}')$ or even $\mathsf{CCP}_\kappa^{\mathsf{Borel}_\kappa}(X, \mathcal{F}')$.

Kechris's games

Kechris introduced a general class of games of length ω which encompasses many of the classical games characterizing dichotomies for subsets of ${}^{\omega}\omega$. We consider the versions of length κ for subsets of the κ-Baire space.

Aim: Obtain their determinacy from CCP.

The κ-perfect set game

Let $X \subseteq {}^{\kappa}2$. $G^*_\kappa(X)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		i_0		i_1		$\dots$		i_α		$\dots$

where $t_\alpha \in {}^{<\kappa}2$ and $i_\alpha < 2$. $\mathbf{I}$ wins if $\bigoplus_{\alpha<\kappa} t_\alpha := t_0{}^\frown t_1{}^\frown \dots t_\alpha{}^\frown \dots$ is in X and $t_{\alpha+1}(0) = i_\alpha$ for all $\alpha < \kappa$.

The κ-perfect set game

Let $X \subseteq {}^{\kappa}2$. $G^*_\kappa(X)$ is the following game of length κ:

$$\begin{array}{ccccccccccc} \mathbf{I} & t_0 & & t_1 & & \dots & & t_\alpha & & \dots & \\ \mathbf{II} & & i_0 & & i_1 & & \dots & & i_\alpha & & \dots \end{array}$$

where $t_\alpha \in {}^{<\kappa}2$ and $i_\alpha < 2$. $\mathbf{I}$ wins if $\bigoplus_{\alpha<\kappa} t_\alpha := t_0{}^\frown t_1{}^\frown \dots t_\alpha{}^\frown \dots$ is in X and $t_{\alpha+1}(0) = i_\alpha$ for all $\alpha < \kappa$.

Theorem (Kovachev)

- $\mathbf{I}$ *wins* $G^*_\kappa(X) \iff X$ *has a* κ*-perfect subset.*
- $\mathbf{II}$ *wins* $G^*_\kappa(X) \iff |X| \leq \kappa$.

Hence $G^*_\kappa(X)$ *is determined* $\iff \mathrm{PSP}_\kappa(X)$ *holds.*

Let $\mathcal{U}({}^{<\kappa}d)$ denote the set of upwards closed subsets of ${}^{<\kappa}d$. Let $X \subseteq {}^{\kappa}d$.

Let R be a nonempty set (requirements) and $S : R \to \mathcal{U}({}^{<\kappa}d)$.
$G^S_\kappa(X)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		r_0		r_1		$\dots$		r_α		$\dots$

where $t_\alpha \in {}^{<\kappa}d$ and $r_\alpha \in R$. **I** wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in S(r_\alpha)$ for all $\alpha < \kappa$.

Kechris's games

Let $\mathcal{U}({}^{<\kappa}d)$ denote the set of upwards closed subsets of ${}^{<\kappa}d$. Let $X \subseteq {}^{\kappa}d$.

Let R be a nonempty set (requirements) and $S : R \to \mathcal{U}({}^{<\kappa}d)$.
$G^S_\kappa(X)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		r_0		r_1		$\dots$		r_α		$\dots$

where $t_\alpha \in {}^{<\kappa}d$ and $r_\alpha \in R$. **I** wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in S(r_\alpha)$ for all $\alpha < \kappa$.

S is nontrivial if for all $i \in d$, there exists $r \in R$ such that $t(0) \neq i$ for all $t \in S(r)$.

Kechris's games

Let $\mathcal{U}({}^{<\kappa}d)$ denote the set of upwards closed subsets of ${}^{<\kappa}d$. Let $X \subseteq {}^{\kappa}d$.

Let R be a nonempty set (requirements) and $S : R \to \mathcal{U}({}^{<\kappa}d)$.
$G_\kappa^S(X)$ is the following game of length κ:

$$\begin{array}{lcccccccccc} \mathbf{I} & t_0 & & t_1 & & \dots & & t_\alpha & & \dots & \\ \mathbf{II} & & r_0 & & r_1 & & \dots & & r_\alpha & & \dots \end{array}$$

where $t_\alpha \in {}^{<\kappa}d$ and $r_\alpha \in R$. $\mathbf{I}$ wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in S(r_\alpha)$ for all $\alpha < \kappa$.

Example

- The κ-perfect set game $G_\kappa^*(X)$ is equivalent to $G_\kappa^S(X)$ for $d := 2$, $R := 2$ and $S(r) := \{t \in {}^{<\kappa}2 : t(0) = r\}$.

Kechris's games

Let $\mathcal{U}({}^{<\kappa}d)$ denote the set of upwards closed subsets of ${}^{<\kappa}d$. Let $X \subseteq {}^{\kappa}d$.

Let R be a nonempty set (requirements) and $S : R \to \mathcal{U}({}^{<\kappa}d)$.
$G^S_\kappa(X)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		r_0		r_1		$\dots$		r_α		$\dots$

where $t_\alpha \in {}^{<\kappa}d$ and $r_\alpha \in R$. **I** wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in S(r_\alpha)$ for all $\alpha < \kappa$.

Example

- The Banach-Mazur game $G^{**}_\kappa(X)$ is equivalent to $G^S_\kappa(X)$ for $d := \kappa$, $R := {}^{<\kappa}\kappa$ and $S(r) := \{t \in {}^{<\kappa}\kappa : r \subseteq t\}$.

Example: the Hurewicz dichotomy

Let R be a nonempty set and $S : R \to \mathcal{U}({}^{<\kappa}\kappa)$. $G^S_\kappa(X)$ is the game:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		r_0		r_1		$\dots$		r_α		$\dots$

I wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in S(r_\alpha)$ for all $\alpha < \kappa$.

Example

Let $R := \kappa$ and $S(r) := \{t \in {}^{<\kappa}\kappa \mid t(0) \geq r\}$. Then $G^S_\kappa(X)$ is equivalent to the κ-superperfect set game, so it's determinacy is equivalent to the following variant of the Hurewicz dichotomy:

$\mathrm{HD}_\kappa(X)$: Either $X \subseteq \bigcup_{\alpha<\kappa}[T_\alpha]$ for $<\kappa$-splitting trees T_α or X contains a κ-superperfect subset (i.e., $[T] \subseteq X$ for a cofinally κ-splitting $<\kappa$-closed tree T.)

Theorem (Sclicht, Sz)

$\mathsf{CCP}_\kappa(X)$ implies that $G^S_\kappa(X)$ is determined for all nonempty sets R of size $\leq \kappa$ and all nontrivial $S : R \to \mathcal{U}({}^{<\kappa}d)$.

Proof sketch.

$A \subseteq {}^{<\kappa}d$ is called S-dense above $t \in {}^{\kappa}d$ if $(\forall r \in R)(\exists s \in S(r))t^\frown s \in A$.

A is S-nowhere dense if it is not S-dense above any $t \in {}^{\kappa}d$.

$\mathcal{F}_S := \{[T] : T$ is an S-nowhere dense subtree of ${}^{<\kappa}d\}$.

Members of $\mathcal{I}_{\mathcal{F}_S}$ are called (κ, S)-meager.

CCP and Kechris's games

Lemma

X is (κ, S)-meager $\iff$ $\mathbf{II}$ has a winning strategy in $G^S_\kappa(X)$.

Proof.

$\Rightarrow$: Suppose $X \subseteq \bigcup_{i<\kappa}[T_i]$ where each T_i is S-nowhere dense. The strategy of $\mathbf{II}$ is to play r_i in each round $i < \kappa$ in such a way that $\mathbf{I}$ is forced to avoid T_i in all further rounds. This is possible since T_i is S-nowhere dense.

$\Rightarrow$: Fix a winning strategy σ for $\mathbf{II}$ and let Run_σ denote the set of those positions $p := \langle t_\beta, r_\beta : \beta < \alpha \rangle$ which follow σ. A position $p \in Run_\sigma$ is good for $x \in X$ if $(\bigoplus_{\beta<\alpha} t_\beta)^\frown s \subseteq x$ for some $s \in S(r_\alpha)$.

$$X_p := \{x \in X : p \text{ is maximal good for } x\}.$$

Then $X = \bigcup_{p \in Run_\sigma} X_p$ and each $T(X_p)$ is S-nowhere dense. □

A strict order preserving map $\phi : {}^{<\kappa}R \to {}^{<\kappa}d$ is S-dense if $\{\phi(t^\frown\langle r\rangle) : r \in R\}$ is S-dense above $\phi(t)$ for all $t \in {}^{<\kappa}R$.

Lemma

The following are equivalent:

1. **I** *has a winning strategy in* $G^S_\kappa(X)$.
2. *There exists an* S*-dense strict order preserving map* $\phi : {}^{<\kappa}R \to {}^{<\kappa}d$ *with* $\mathrm{ran}([\phi]) \subseteq X$,
3. *There exists a continuous map* $f : {}^{\kappa}R \to X$ *with* $f(N_t) \in \mathcal{I}^+_{\mathcal{F}}$ *for all* $t \in {}^{<\kappa}\kappa$.

Hence $\mathsf{CCP}_\kappa(X, \mathcal{F}_S) \iff (G^S_\kappa(X)$ is determined).

□

Proof of the Lemma. (1)⇔(2) and (2)⇒(3) are clear.

(1)⇔(2) is analogous to the corresponding lemma about ABP from CCP: Construct strict order preserving maps $\phi : {}^{<\kappa}R \to {}^{<\kappa}d$ and $\iota : {}^{<\kappa}R \to {}^{<\kappa}R$ and $s^t_r \in S(r)$ for each $t \in {}^{<\kappa}R$ and $r \in R$ such that

- $\phi(t^\frown\langle r\rangle) \supseteq \phi(t)^\frown s^t_r$,
- $f(N_{\iota(t^\frown\langle r\rangle)}) \subseteq N_{\phi(t)^\frown s^t_r}$

and ι is continuous. Then ϕ will be an S-dense map since each $S(r)$ is upwards closed, so $\phi(t^\frown\langle r\rangle) \setminus \phi(t) \in S(r)$. Moreover, $[\phi] = f \circ [\iota]$.

The construction: If $\iota(t)$, $\phi(t)$ and s^v_r have been constructed for all $v \subsetneq t$, let

$$u := \begin{cases} \phi(v)^\frown s^v_r & \text{if } t = v^\frown\langle r\rangle, \\ \bigcup_{s \subsetneq t} \phi(s) & \text{if } \mathrm{lh}(t) \in \mathrm{Lim}. \end{cases}$$

Since $T(f(N_{\iota(t)}))$ is somewhere S-dense, take $\phi(t) \supseteq u$ such that $T(f(N_{\iota(t)}))$ is S-dense above $\phi(t)$. Choose $s^t_r \in S(r)$ for each $r \in R$ so that $\phi(t)^\frown s^t_r \in T(f(N_{\iota(t)}))$, and find $\iota(t^\frown\langle r\rangle)$ using the continuity of f. □

Further applications of CCP

$\mathsf{CCP}_\kappa(X)$ implies each of the following:

- The topological Hurewicz dichotomy: either X is covered by κ-many κ-compact sets, or X contains a closed subset of ${}^\kappa\kappa$ which is homeomorphic to ${}^\kappa\kappa$.

  $\mathcal{F} :=$ {κ-compact sets}.

- The Kechris-Louveau-Woodin dichotomy: For all $Y \subseteq {}^\kappa\kappa \setminus X$, either there is a $\mathbf{\Sigma}^0_2$ set separating X from Y, or $X \cup Y$ contains a closed subset C which is homeomorphic to ${}^\kappa 2$ such that $C \cap Y$ is a dense subset of C of size κ.

  $\mathcal{F} :=$ {closed sets which are disjoint from Y}.

- The open graph dichotomy: for any open graph G on $X \times X$, either G admits a κ-coloring, or G has a κ-perfect complete subgraph.

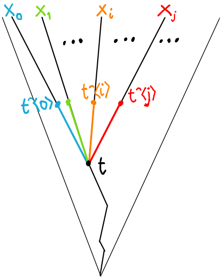
  $\mathcal{F} :=$ {closed G-independent sets}.

The open dihypergraph dichotomy

Aim: $\mathsf{CCP}_\kappa(X)$ is equivalent to a generalization of the open graph dichotomy for infinite dimensional directed dihypergraphs.

The open dihypergraph dichotomy

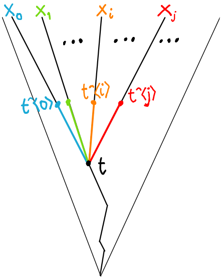
A d-dihypergraph on a set X is a set of nonconstant sequences in ${}^{d}X$. Fix the box topology on ${}^{d}X$ with basic open sets $\prod_{i \in d} U_i$, where each U_i is open in X.



$\mathsf{ODD}^{d}_{\kappa}(X)$: For all box-open d-dihypergraphs H on X, either H admits a κ-coloring, or there is a continuous homomorphism $f : {}^{\kappa}d \to X$ from $\mathbb{H}_{{}^{\kappa}d} := \bigcup_{t \in {}^{<\kappa}d} \prod_{i \in d} N_{t^\frown\langle i\rangle}$ to H.

The open dihypergraph dichotomy

A d-dihypergraph on a set X is a set of nonconstant sequences in ${}^dX$. Fix the box topology on ${}^dX$ with basic open sets $\prod_{i\in d} U_i$, where each U_i is open in X.

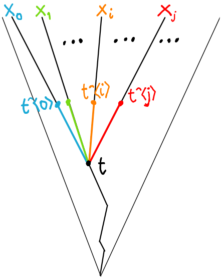


$\mathsf{ODD}^d_\kappa(X)$: For all box-open d-dihypergraphs H on X, either H admits a κ-coloring, or there is a continuous homomorphism $f : {}^\kappa d \to X$ from $\mathbb{H}_{{}^\kappa d} := \bigcup_{t\in {}^{<\kappa}d} \prod_{i\in d} N_{t^\frown\langle i\rangle}$ to H.

$\mathsf{ODD}^d_\kappa(X, \mathsf{Def}_\kappa)$ denotes the restriction to definable box-open dihypergraphs.

The open dihypergraph dichotomy

A d-dihypergraph on a set X is a set of nonconstant sequences in ${}^dX$. Fix the box topology on ${}^dX$ with basic open sets $\prod_{i\in d} U_i$, where each U_i is open in X.



$\mathrm{ODD}^d_\kappa(X)$: For all box-open d-dihypergraphs H on X, either H admits a κ-coloring, or there is a continuous homomorphism $f : {}^\kappa d \to X$ from $\mathbb{H}_{{}^\kappa d} := \bigcup_{t\in {}^{<\kappa}d} \prod_{i\in d} N_{t^\frown\langle i\rangle}$ to H.

$\mathrm{ODD}^d_\kappa(X, \mathrm{Def}_\kappa)$ denotes the restriction to definable box-open dihypergraphs.

Theorem (Schlicht, Sz 2023)

After a Lévy collapse of λ to κ^+, the following hold for all definable subsets X of ${}^\kappa\kappa$:

- $\mathrm{ODD}^\kappa_\kappa(X)$ *if λ is Mahlo.*
- $\mathrm{ODD}^\kappa_\kappa(X, \mathrm{Def}_\kappa)$ *if λ is inaccessible.*

Theorem (Sclicht,Sz)

For any $X \subseteq {}^{\kappa}\kappa$:

- $\mathsf{CCP}_\kappa(X) \iff \mathsf{ODD}_\kappa^\kappa(X)$.
- $\mathsf{CCP}_\kappa(X, \mathsf{Def}_\kappa) \iff \mathsf{ODD}_\kappa^\kappa(X, \mathsf{Def}_\kappa)$.

Hence $\mathsf{CCP}_\kappa(X)$ holds for all definable sets X after a Lévy-collapse of a Mahlo cardinal, and an inaccessible suffices for $\mathsf{CCP}_\kappa(X, \mathsf{Def}_\kappa)$.

Form CCP to ODD

Proof sketch.

Lemma

Suppose H is a box-open κ-dihypergraph on X. Let $\mathcal{F}$ be the family of all closed H-independent subsets of ${}^\kappa\kappa$.

1. $Y \in \mathcal{I}_\mathcal{F} \iff H{\restriction}Y$ *has a* κ*-coloring, for all* $Y \subseteq X$.
2. *The existence of the following objects is equivalent:*
   - *a continuous homomorphism from* $\mathbb{H}_{{}^\kappa\kappa}$ *to* H,
   - *a continuous map* $f : {}^\kappa\kappa \to X$ *with* $f(N_t) \notin \mathcal{I}_\mathcal{F}$ *for all* $t \in {}^{<\kappa}\kappa$.

Hence $\mathsf{CCP}^{\mathcal{F}}_\kappa(X) \iff \mathsf{ODD}^{H\restriction X}_\kappa$.

Proof of (2) in the Lemma.

$\Downarrow$: Suppose $f : {}^{\kappa}\kappa \to X$ is a continuous homomorphism from $\mathbb{H}_{{}^{\kappa}\kappa}$ to X.

Claim. $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$.

Proof. Suppose $f(N_t) \subseteq \bigcup_{\alpha<\kappa} X_\alpha$ where each $X_\alpha \in \mathcal{F}$. Construct a continuous increasing sequence $\langle t_\alpha : \alpha < \kappa \rangle$ such that $t_0 = t$ and for each $\alpha < \kappa$,

- $t_{\alpha+1}$ is an immediate successor of t_α
- $f(N_{t_{\alpha+1}}) \cap X_\alpha = \emptyset$.

This is possible since otherwise, there exists x_i in $f(N_{t_\alpha \frown \langle i \rangle}) \cap X_\alpha$ for each $i < \kappa$. Since f is a homomorphism, $\langle x_i : i < \kappa \rangle \in H \restriction X_\alpha$. So $X_\alpha \notin \mathcal{F}$. □

Form CCP to ODD

Proof of (2) in the Lemma.

$\Uparrow$: Suppose $f : {}^{\kappa}\kappa \to X$ is continuous with $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$. Construct continuous strict order preserving maps $\phi, \iota : {}^{<\kappa}\kappa \to {}^{<\kappa}\kappa$ such that for all $t \in {}^{<\kappa}\kappa$,

- $\prod_{i<\kappa} N_{\phi(t^\frown\langle i\rangle)} \cap X \subseteq H$,
- $f(N_{\iota(t)}) \subseteq N_{\phi(t)}$.

Then $[\phi] = f^\frown[\iota]$ will be a continuous homomorphism from $\mathbb{H}_{\kappa_d}$ to H. □

Form ODD to CCP

Let $\mathcal{F}$ be a family of closed subsets of ${}^\kappa\kappa$. We may assume $\mathcal{I}_\mathcal{F} \cap \mathbf{\Pi}^0_1 = \mathcal{F}$.

Let H consist of all κ-sequences $\langle x_\alpha : \alpha < \kappa\rangle \in {}^\kappa\kappa$ with $\overline{\{x_\alpha : \alpha < \kappa\}} \notin \mathcal{F}$.

Lemma

A closed subset C of ${}^\kappa\kappa$ is H-independent if and only if $C \in F$.

Proof. $\Rightarrow$: Take a κ-sequence $\langle x_\alpha : \alpha < \kappa\rangle \in {}^\kappa\kappa$ whose range is dense in C. Then $C = \overline{\{x_\alpha : \alpha < \kappa\}}$ is not in $\mathcal{F}$, so it is $\mathcal{I}_\mathcal{F}$-positive.

$\Leftarrow$: If $H{\upharpoonright}C$ has a hyperedge $\langle x_\alpha : \alpha < \kappa\rangle \in {}^\kappa\kappa$ then $C \notin \mathcal{F}$ since C is a superset of the $\mathcal{I}_\mathcal{F}$-positive set $\overline{\{x_\alpha : \alpha < \kappa\}}$. □

Hence $\mathsf{CCP}^\mathcal{F}_\kappa(X) \iff \mathsf{ODD}^{H{\upharpoonright}X}_\kappa$ by the previous slide. □

Open questions

Question

What is the consistency strength of $\mathsf{CCP}_\kappa(X)$ for all definable subsets $X \subseteq {}^\kappa\kappa$?

It is at least an inaccessible cardinal. A Mahlo cardinal is needed for the proof.

Question

Is $\mathsf{CCP}_\kappa(X)$ equivalent to $\mathsf{CCP}_\kappa^{\mathsf{C}}(X)$ for $\mathsf{C} = \mathbf{\Sigma}_1^1$ or $\mathsf{C} = \mathsf{C}_\kappa$ when $\kappa > \omega$?

In the countable setting, DiPrisco and Todorčević have obtained some further applications of CCP, such as:

- $\mathsf{CCP}_\omega(X)$ implies the Transitive Coloring Axiom in dimensions $n < \omega$: any open reflexive transitive n-hypergraph on X has either a coutable coloring or a perfect complete subgraph.
- For every non-atomic P-ideal $\mathcal{I}$ on ω, $\mathsf{CCP}_\omega(\mathcal{I})$ implies that there is a monotonic map from $\mathcal{I}$ into a cofinal subset of ${}^\omega\omega$ ordered by *everywhere* dominance.

Can these be lifted to the uncountable setting?

Thank you!