$\Sigma_1(X)$ and $\Pi_1(X)$ definability above large cardinals

Farmer Schlutzenberg, TU Vienna

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Reference:

- Low level definability above large cardinals, arXiv:2401.01979

Background:

- Lücke and Schlicht, Measurable cardinals and good $\Sigma_1(\{\kappa\})$ wellorderings,
- Lücke, Schindler and Schlicht, $\Sigma_1(\{\kappa\})$-definable subsets of H_{κ^+}
- Väänänen and Welch, When cardinals determine the power set: inner models and Härtig quantifier logic.
- Lücke and Müller, Σ_1-definability at higher cardinals: Thin sets, almost disjoint families and long well-orders

Suppose there are large cardinals $\leq \kappa$.

For this talk, consider definability of:

- wellorders of subsets of $\mathcal{P}(\kappa)$ of cardinality $> \kappa$,
- good wellorders of $\mathcal{P}(\kappa)$,
- ultrafilters over κ and club filter at κ.

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- wellorders of subsets of $\mathcal{P}(\kappa)$ of cardinality $> \kappa$,
- good wellorders of $\mathcal{P}(\kappa)$,
- ultrafilters over κ and club filter at κ.
- (slides also contain some things about almost disjoint families and independent families).

Focus: $\Sigma_1(X)$ and $\Pi_1(X)$ definability, for various X.

Definability

Definition 0.1.

Let C, X classes.

C is $\Sigma_1(X)$-definable or just C is $\Sigma_1(X)$ iff there is a Σ_1 formula φ and $\vec{x} \in X^{<\omega}$ such that for all y,

$$y \in C \iff \varphi(y, \vec{x}).$$

Likewise C is $\Pi_1(X)$.

C is $\Delta_1(X)$ iff C is both $\Sigma_1(X)$ and $\Pi_1(X)$.

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Definition 0.2.

$\mathcal{H}_\kappa$ is the set of sets hereditarily of cardinality $< \kappa$.
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Question 0.3.

Which kinds of sets $C \subseteq \mathcal{H}_{\kappa^+}$ can be $\Sigma_1(X)$ or $\Pi_1(X)$, for a given X?

E.g. $X = \mathcal{H}_\kappa \cup \mathrm{OR}$ (where OR is the ordinals).

Let W be a wellorder a set D. For $x \in D$, define

$$D_x = \{y \in D \mid yWx\}$$

and the <u>initial segment of W below x</u>

$$W_x = W \restriction D_x.$$

Recall:

Definition 0.4.

W is called a <u>$\Sigma_1(X)$-good wellorder</u> iff

$$\{(x, W_x) \mid x \in D\}$$

is $\Sigma_1(X)$.

If W is a $\Sigma_1(X)$-good wellorder of $\mathcal{H}_{\kappa^+}$ where $X \subseteq \mathcal{H}_{\kappa^+}$, then W has ordertype κ^+.

Definition 0.5.

If $\mu < \kappa$ are cardinals, say κ is μ-steady iff there is a cardinal ν such that:

- $\kappa = \nu$ or $\kappa = \nu^+$,
- $\mathrm{cof}(\nu) \neq \mu$,
- ν is μ-closed; that is, $\alpha^\mu < \nu$ for all $\alpha < \nu$.

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Under GCH, if μ is measurable and $\mu < \kappa$, then TFAE:

- κ is μ-steady,
- $\mathrm{cof}(\kappa) \neq \mu$ and κ is not the successor of a cardinal ν with $\mathrm{cof}(\nu) = \mu$,
- $j(\kappa) = \kappa$, for some/all measures on μ.

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Definition 0.6.

A measure on a cardinal μ just means a μ-complete non-principal ultrafilter on μ.

Recall $L[U]$ is a proper class inner model for 1 measurable cardinal.

Theorem 0.7 (Lücke, Schlicht).

Assume $V = L[U]$ where μ is measurable. Let κ be an uncountable cardinal. Then there is a $\Sigma_1(\{\kappa\})$-good wellorder of $\mathcal{P}(\kappa)$ iff either:

- $\kappa \leq \mu^+$*, or*
- κ *is non-*μ*-steady.*

(They also generalized this to $L[U_0, U_1]$.)

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Question 0.8 (Lücke, Schlicht).

Assume $V = L[U]$. Let $\kappa > \mu^+$ be a μ-steady cardinal.
Is there a $\Sigma_1(\{\kappa\})$ wellorder of $\mathcal{P}(\kappa)$?

Lücke and Müller answered this question...

Theorem 0.9 (Lücke, Müller).

Suppose μ is measurable and $\kappa > \mu^+$ is μ-steady, as witnessed by $\nu \leq \kappa$. Then:

(i) There is no $\Sigma_1(V_\mu \cup \{\nu, \nu^+\})$ wellorder of $\mathcal{P}(\kappa)$.

(ii) If $\mathrm{cof}(\kappa) > \omega$ then there is no $\Sigma_1(V_\mu \cup \{\kappa\})$ injection $f : \kappa^+ \to \mathcal{P}(\kappa)$.

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A refinement:

Theorem 0.10 (S.).

Let μ be measurable and $\kappa > \mu^+$ be a μ-steady cardinal. Let U be a measure on μ. Then:

(i) There is no $\Sigma_1(S_\omega)$ wellorder of $[\eta]^\omega_\uparrow$ (and hence no such wellorder of $\mathcal{P}(\kappa)$).

(ii) There is no $\Sigma_1(S_\kappa)$ injective function $f : \kappa^+ \to \mathcal{P}(\kappa)$.

(iii) For any $\Sigma_1(S_\kappa)$ set $f \subseteq \kappa^+ \times \mathcal{P}(\kappa)$ and any club $C \subseteq \kappa^+$, $f \restriction C$ is not an injective function,

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(iii) For any $\Sigma_1(S_\kappa)$ set $f \subseteq \kappa^+ \times \mathcal{P}(\kappa)$ and any club $C \subseteq \kappa^+$, $f \upharpoonright C$ is not an injective function,

where the following holds: Let

$$j : V \to \mathrm{Ult}(V, U)$$

the ultrapower. Then η is the supremum of the critical sequence of j.
And for $\delta \in \mathrm{OR}$, S_δ is the class of U-$[0, \delta)$-stable sets.

There is no definability requirement on C.

Giving those definitions...

Definition 0.11.

For an ordinal η, $[\eta]^{\omega}_{\uparrow}$ denotes $\{A \subseteq \eta \mid A \text{ has ordertype } \omega\}$.

Definition 0.12.

Recall that the <u>critical sequence of j</u> is the sequence $\langle \mu_n \rangle_{n<\omega}$ where $\mu_0 = \mu = \mathrm{cr}(j)$, and $\mu_{n+1} = j(\mu_n)$.

So $\mu_n < \mu_{n+1}$ for all n.

Definition 0.13 (Recall...).

Let U be a measure on measurable cardinal μ.
Let $\mathcal{T}$ be the iteration of V given by using U and its images. That is, we define M_α, U_α, and $i_{\alpha\beta} : M_\alpha \to M_\beta$ for $\alpha \leq \beta$, as follows:

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$$i_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$$

is the ultrapower map, and $U_{\alpha+1} = i_{\alpha,\alpha+1}(U_\alpha)$.
And for $\beta < \alpha$, we define

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(iv) Given M_α for all $\alpha < \lambda$, where λ is a limit, and given $i_{\alpha\beta} : M_\alpha \to M_\beta$ for all $\alpha < \beta < \lambda$, then M_λ is the direct limit under these maps, and

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We write $M^{\mathcal{T}}_\alpha = M_\alpha$, $i^{\mathcal{T}}_{\alpha\beta} = i_{\alpha\beta}$, etc.
Write $\mathcal{T}_U = \mathcal{T}$.

Definition 0.14.

Let $U, \mathcal{T}, M_\alpha$, etc, be as above.
For $\gamma \leq \delta$ ordinals, say a set x is U-$[\gamma, \delta)$-stable iff

$$i_{\gamma\alpha}(x) = x \text{ for all } \alpha \text{ such that } \gamma \leq \alpha < \delta.$$

Just say $[\gamma, \delta)$-stable if U is clear.

We will prove part (ii), i.e. that there is no $\Sigma_1(S_\kappa)$ injection $f : \kappa^+ \to \mathcal{P}(\kappa)$.
First a standard fact:

Lemma 0.15.

Let U be a measure and

$$i_{\alpha\beta} : M_\alpha \to M_\beta$$

the iteration maps of $\mathcal{T}_U$.

Let κ be a limit ordinal and ξ any ordinal.

Then there is $\alpha < \kappa$ such that ξ is U-$[\alpha, \kappa)$-stable; i.e.

$$i_{\alpha\beta}(\xi) = \xi \text{ for all } \beta \in [\alpha, \kappa).$$

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Remark 0.16.

If $\kappa > \mu$ is μ-steady, where μ is measurable, U is a measure on μ, then κ and κ^+ are U-$[0, \kappa)$-stable.

Proof of part (ii) of theorem.

Suppose some injection

$$f : \kappa^+ \to \mathcal{P}(\kappa)$$

is defined by a Σ_1 formula φ and $\vec{p} \in (S_\kappa)^{<\omega}$, so for $\alpha < \kappa^+$ and $A \subseteq \kappa$,

$$f(\alpha) = A \iff \varphi(\vec{p}, \alpha, A).$$

Let $\mathcal{T} = \mathcal{T}_U$ (recall U is a measure on μ). Let M_α be the αth model and

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the iteration map. So for all $\alpha < \kappa$,

$$i_{0\alpha}(\vec{p}) = \vec{p}$$

and

$$i_{0\alpha}(\kappa) = \kappa \text{ and } i_{0\alpha}(\kappa^+) = \kappa^+$$

Since

$$V \models \text{“}f : \kappa^+ \to \mathcal{P}(\kappa) \text{ is defined by } \varphi(\vec{p}, \cdot, \cdot)\text{”},$$

and $i_{0\alpha}$ fixes the relevant parameters, M_α satisfies the same statement.

The resulting function defined in M_α is just f, by Σ_1 upwards absoluteness. □

Proof continued...

Given $\xi < \kappa^+$, we can fix $\alpha < \kappa$ such that ξ is $[\alpha, \kappa)$-stable. Note that

$$i_{\alpha\beta}(f(\xi)) = f(\xi)$$

for all $\beta \in [\alpha, \kappa)$. Let

$$f(\xi)^* = i_{\alpha\kappa}(f(\xi))$$

for (any/all) such α. So $f(\xi)^* \in M_\kappa$.

Claim: $f(\xi)^* \cap \kappa = f(\xi)$.

Proof: Let $\gamma < \kappa$. Let $\alpha \in (\gamma, \kappa)$ with

$$f(\xi)^* = i_{\alpha\kappa}(f(\xi)).$$

Then

$$\gamma \in f(\xi) \iff i_{\alpha\kappa}(\gamma) \in i_{\alpha\kappa}(f(\xi)) \iff \gamma \in f(\xi)^*,$$

giving the claim; the rightmost equivalence is because

$$\gamma = i_{\alpha\kappa}(\gamma) \text{ and } i_{\alpha\kappa}(f(\xi)) = f(\xi)^*.$$

So $f(\xi) \in M_\kappa$ for all $\xi < \kappa^+$, but $\mathcal{P}(\kappa) \cap M_\kappa$ has cardinality κ, contradiction. □

Theorem 0.17 (Lücke, Müller).

Let κ be a limit a of measurable cardinals. Then:

(i) No wellorder of $\mathcal{P}(\kappa)$ is $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa, \kappa^+\})$.

Limits of measurables

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(iii) *Suppose:*

- $\operatorname{cof}(\kappa) = \omega$,
- $D \subseteq \mathcal{P}(\kappa)$ *has cardinality* $> \kappa$,
- W *is a wellorder of* D,
- D, W *are* $\Sigma_1(\{\kappa\})$.

Then there is a $\utilde{\Sigma}^1_3$ wellorder of the reals.

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Question 0.18 (Lücke, Müller).

Is the hypothesis of (iii) consistent? What if $\mathrm{cof}(\kappa) > \omega$?

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No...

Theorem 0.19 (S.).

Assume κ is a limit of measurables. Then:

(a) No injection $f : \kappa^+ \to \mathcal{P}(\kappa)$ is $\Sigma_1(\mathcal{H}_\kappa \cup \mathrm{OR})$.

(b) There are no D, W such that:

- *$D \subseteq \mathcal{P}(\kappa)$ has cardinality $> \kappa$,*
- *W is a wellorder of D,*
- *D, W are both $\Sigma_1(\mathcal{H}_\kappa \cup \mathrm{OR})$.*

(Cofinality of κ is irrelevant. In (a), no such f can be injective on a club $C \subseteq \kappa^+$.)

Question 0.20.

Is it consistent that:

- there are (more) large cardinals,
- for every κ which is not a limit of measurables, there is a $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$ wellorder of $\mathcal{P}(\kappa)$?
- Or even a $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$-good wellorder?

(A partial answer coming...)

Motivation for question:

- Lücke, Schlicht: characterized those κ such that there is a $\Sigma_1(\{\kappa\})$-good wellorder of $\mathcal{P}(\kappa)$ in $L[U]$.

  (For some κ, there isn't.)
- Lücke, Müller: No $\Sigma_1(\mathcal{H}_\mu \cup \{\nu, \nu^+\})$ wellorder of $\mathcal{P}(\kappa)$ when μ measurable $< \kappa$ and κ is μ-steady.
- Lücke, Müller: No $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$ wellorder of $\mathcal{P}(\kappa)$ when κ a limit of measurables.

Mice:

- Structures M of form $L[\mathbb{E}]$ or $L_\alpha[\mathbb{E}]$ where $\mathbb{E}$ is a sequence of extenders $\mathbb{E}$ (like measures).
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- From $\mathbb{E}^M$ can determine order of constructibility $<^M$, a global wellorder of universe of M.
- For all cardinals θ of M, $\mathcal{H}_\theta^M$ is an initial segment of $<^M$.
- Good wellorders of $\mathcal{P}(\kappa)^M$ defined in mice M are typically a restriction of $<^M$.

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- Can use extenders from $\mathbb{E}$ to form ultrapowers
- Some $E \in \mathbb{E}$ are partial
- Iterations have non-linear structure
- Iteration strategies guide the formation of iterations.
- Can have many Woodin cardinals.
- Roughly, the "mouse order" is wellfounded and ranks mice according to large cardinals.
- From $\mathbb{E}^M$ can determine order of constructibility $<^M$, a global wellorder of universe of *M*.
- For all cardinals θ of *M*, $\mathcal{H}_\theta^M$ is an initial segment of $<^M$.
- Good wellorders of $\mathcal{P}(\kappa)^M$ defined in mice *M* are typically a restriction of $<^M$.
- If $\mathbb{E}^M \restriction \kappa^{+M}$ is simply $\Sigma_1(X)$ over *M*, then $<^M$ yields a $\Sigma_1(X)$-good wellorder of $(\mathcal{H}_{\kappa^+})^M$.

M_1 is the canonical proper class mouse with 1 Woodin cardinal.

Theorem 0.21 (Steel).

Work in $M_1 = L[\mathbb{E}]$. Then $\mathbb{E} = \mathbb{E}^K$, where K is the "core model". Therefore $\mathbb{E}$ is definable over the universe of M_1 without parameters.

A related result:

Theorem 0.22 (S., approx 2007, 2014).

Let M be a mouse with no largest cardinal, which is sufficiently self-iterable (can define enough of its own iteration strategy). Then $\mathbb{E}^M$ is definable over the universe of M without parameters.

For M_1, Welch and Väänänen show:

Theorem 0.23 (Welch, Väänänen, 2021).

In M_1, the relation "$x = \mathcal{P}(y)$" is Σ_1-in-Card (Card is allowed as a predicate).

(They show that $\mathbb{E}^{M_1}$ is Σ_1-in-Card.)

All 3 come down to identifying $\mathbb{E}^M$ via comparison and universality arguments, and require strong degree of self-iterability.

A somewhat different kind of proof (based more on condensation) gives:

Theorem 0.24 (S., approx 2015).

Let M be a mouse and $\kappa > \omega_1^M$ an uncountable cardinal of M. Then $\mathbb{E}\restriction\kappa$ is definable over $\mathcal{H}_\kappa^M$ from the parameter $\mathbb{E}\restriction\omega_1^M$.

This doesn't require any self-iterability.

Question 0.25.

In a mouse $M = L[\mathbb{E}]$, how simply definable are $\mathbb{E}^M$ and $<^M$?

For example:

- Is $\mathbb{E} \restriction \kappa^+$ a $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$-definable set? (By L-M, not if κ limit of meas.)
- Is $<^M \restriction \kappa^+$ a $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$-definable set?
- If yes, then $\{\mathbb{E} \restriction \kappa\}$ and $\{\mathcal{H}_\kappa\}$ are also.

Theorem 0.26 (S.).

Work in $M_1 = L[\mathbb{E}]$. Let κ be an uncountable cardinal which is not a limit of measurables, and not Mahlo. Then:

(i) $\{\mathbb{E}|\kappa\}$ *is* $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$,

(ii) $\{\mathcal{H}_\kappa\}$ *is* $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$.

But some smallness is necessary:

Theorem 0.27.

Let κ be ω_1-iterable. Then $\{V_\kappa\} = \{\mathcal{H}_\kappa\}$ is not $\Sigma_1(V_\kappa \cup \{\kappa\})$.

Measurable > ω_1-iterable > weakly compact.

Theorem 0.28.

Work in M_1. For all uncountable cardinals κ which are not a limit of measurables, there is a $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$-good wellorder of $\mathcal{H}_{\kappa^+}$, and also of $\mathcal{P}(\kappa)$.

Remark 0.29.

– If κ is ω_1-iterable, it can't just be the restriction of $<^M$.

– What if not a restriction of $<^M$?

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In the proof, the good wellorder is lexicographic:

- first on mouse order,
- then order of constructibility (in an identified mouse).

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- first on mouse order,
- then order of constructibility (in an identified mouse).

Given $X, Y \subseteq \kappa$, we first identify the "least good mouse" N_X with $X \in N_X$, and likewise N_Y, and then set $X <^* Y$ iff:

- N_X is strictly below N_Y in the mouse order, or
- $N_X = N_Y \models$"$X <^{N_X} Y$" (where $<^{N_X}$ is the constructibility order of N_X).

($\mathcal{H}_\kappa$ will not be an initial segment of the order for $\mathcal{H}_{\kappa^+}$.)

Theorem 0.30.

Let κ be a limit of measurable cardinals with $\text{cof}(\kappa) = \omega$. Then there is no $\Sigma_1(\mathcal{H}_\kappa \cup \mathrm{OR})$ ultrafilter over κ which contains no bounded subsets of κ.

Proof sketch.

Suppose not and let φ be Σ_1 and $\vec{p} \subseteq \mathcal{H}_\kappa \cup \mathrm{OR}$ be such that

$$U = \{A \mid \varphi(\vec{p}, A)\}$$

is an ultrafilter over κ. For simplicity assume $\vec{p} \subseteq \mathcal{H}_\kappa \cup \{\kappa\}$.

Fix a strictly increasing sequence $\langle \mu_n \rangle_{n<\omega}$ of measurables with supremum κ.
Fix a sequence $\langle D_n \rangle_{n<\omega}$ of measures D_n on μ_n, with

$$\vec{p} \cap \mathcal{H}_\kappa \subseteq \mathcal{H}_{\kappa_0}.$$

Let $A = \bigcup_{n<\omega}[\kappa_{2n}, \kappa_{2n+1})$ and $B = \bigcup_{n<\omega}[\kappa_{2n+1}, \kappa_{2n+2})$.
Then

$$A \in U \iff B \notin U.$$

Proof continued...

Suppose $A \in U$.

Consider the iteration $\mathcal{T}$ using measures from $\{D_n\}_{n<\omega}$ and their pointwise images, of length $\kappa + 1$, with final iteration map $\ell = i_{0\kappa}$,

$$\ell : V \to M_\kappa$$

such that

$$\ell(\kappa_n) = \kappa_{n+1}$$

for all $n < \omega$.

Now observe that:

- $\ell(\kappa) = \kappa$,
- $\ell(\vec{p}) = \vec{p}$,
- $\ell(A) = B$.

Since $\varphi(\vec{p}, A)$ holds, by elementarity, $M_\kappa \models \varphi(\ell(\vec{p}), \ell(A))$, so

$$M_\kappa \models \varphi(\vec{p}, B).$$

But φ is Σ_1, so then $\varphi(\vec{p}, B)$ really holds, so $B \in U$, contradiction. □

Let Club_κ denote the club filter at κ. Some facts:

- (S. Friedman, L. Wu) If κ is weakly compact then Club_κ is not $\Pi_1(\mathcal{H}_{\kappa^+})$.
- (Lücke, Schindler, Schlicht) If κ is a regular cardinal which is a stationary limit of ω_1-iterable cardinals then Club_κ is not $\Pi_1(\{\kappa\})$.

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Theorem 0.31.

Let κ be inaccessible. Then:

(i) Suppose $\mu < \kappa$ and μ is measurable, as witnessed by a measure U on μ. Let

$$j : V \to \mathrm{Ult}(V, U)$$

be the ultrapower map and $S_j = \{x \mid j(x) = x\}$.

Then Club_κ is not $\Pi_1(S_j)$.

(ii) Suppose κ is a limit of measurables. Then Club_κ is not $\Pi_1(V_\kappa \cup \mathrm{OR})$.

Let Club_κ denote the club filter at κ. Some facts:

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(ii) Suppose κ is a limit of measurables. Then Club_κ is not $\Pi_1(V_\kappa \cup \mathrm{OR})$.

Part (ii) is not just a corollary to the facts mentioned above: If κ is the least inaccessible limit of measurables, then the set C of all limits of measurables $< \kappa$ is club and consists of singular cardinals. But every ω_1-iterable is weakly compact.

Proof.

Part (i): Suppose otherwise.

CLAIM 1.

Let $A \subseteq \kappa$ be such that $M = \mathrm{Ult}(V, U) \models$"$A$ is stationary". Then A is stationary.

Proof.

Fix a Π_1 formula φ and $\vec{x} \in (S_j)^{<\omega}$ such that for all $A \subseteq \kappa$,

$$A \in \mathrm{Club}_\kappa \iff \varphi(\vec{x}, A).$$

Then stationarity in κ is $\Sigma_1(S_j)$, and in fact for $A \subseteq \kappa$,

$$A \text{ is stationary} \iff \kappa \backslash A \notin \mathrm{Club}_\kappa \iff \neg\varphi(\vec{x}, \kappa \backslash A).$$

By elementarity of j and since $j(\vec{x}) = \vec{x}$ and $j(\kappa) = \kappa$, M satisfies this same characterization of stationarity.

Proof.

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By elementarity of j and since $j(\vec{x}) = \vec{x}$ and $j(\kappa) = \kappa$, M satisfies this same characterization of stationarity.

But then if $A \subseteq \kappa$ and $A \in M \models$"A is stationary", then

$$M \models \neg\varphi(\vec{x}, \kappa\backslash A),$$

but $\neg\varphi$ is Σ_1, so

$$\neg\varphi(\vec{x}, \kappa\backslash A),$$

so A is stationary. □

Proof continued...

We now exhibit a non-stationary set $A \in M$ such that $M \models$"A is stationary".
Let

$$A = \{\alpha < \kappa \mid \mathrm{cof}^M(\alpha) = \mu^+\}.$$

Then $M \models$"A is stationary".

Claim 2.

A is non-stationary.

Proof.

Let C be the closure of $j``\kappa$ in κ. So C is club in κ, so it suffices to see that

$$C \cap A = \emptyset.$$

Since $\mu^+ \notin \mathrm{rg}(j)$, we have

$$A \cap (j``\kappa) = \emptyset,$$

so it suffices to see that

$$A \cap (C \setminus (j``\kappa)) = \emptyset.$$

Let $\alpha \in C \setminus (j``\kappa)$. Then $\mathrm{cof}(\alpha) = \mu$.
But M is closed under μ-sequences, so $\mathrm{cof}^M(\alpha) = \mu$ also, so $\alpha \notin A$. □

This completes the proof. □

Proof.

Part (ii): This is an immediate corollary of part (i) and the theorem of Kunen below. □

Theorem 0.32 (Kunen).

Let α be an ordinal. Then there are only finite many measurable cardinals μ such that $j(\alpha) > \alpha$, where U is some measure on μ and

$$j : V \to \mathrm{Ult}(V, U)$$

is the ultrapower map.

Almost disjoint families

Definition 0.33.

Let κ be an infinite cardinal. Recall an almost disjoint family (at κ) is a set

$$\mathscr{F} \subseteq \mathcal{P}(\kappa)$$

such that:

- A is unbounded in κ for every $A \in \mathscr{F}$, and
- $A \cap B$ is bounded in κ for all $A, B \in \mathscr{F}$ with $A \neq B$.

And $\mathscr{F}$ maximal if there is no almost disjoint $\mathscr{F}'$ with $\mathscr{F} \subsetneq \mathscr{F}'$.
Mad = maximal almost disjoint.

Theorem 0.34 (Mathias).

There is no $\underset{\sim}{\Sigma}^1_1$ infinite mad family at ω.

Theorem 0.35 (Lücke, Müller).

If κ is an ω_1-iterable cardinal which is a limit of measurables then there is no $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$ almost disjoint family $\mathscr{F}$ at κ such that $\mathscr{F}$ has cardinality $> \kappa$.

(Thus, no $\Sigma_1(\mathcal{H}_\kappa \cup \{\kappa\})$ mad family of cardinality $\geq \kappa$.)

Theorem 0.36 (S.).

If κ a limit of measurable cardinals then:

- *there is no $\Sigma_1(\mathcal{H}_\kappa \cup \mathrm{OR})$ mad family $\mathscr{F} \subseteq \mathcal{P}(\kappa)$ of cardinality $\geq \kappa$, and*
- *if $\mathrm{cof}(\kappa) > \omega$ then there is no $\Sigma_1(\mathcal{H}_\kappa \cup \mathrm{OR})$ almost disjoint family $\mathscr{F} \subseteq \mathcal{P}(\kappa)$ of cardinality $> \kappa$.*

(Exercise: If $\mathrm{cof}(\kappa) = \omega$ then there is a $\Delta_1(\{\kappa\})$ almost disjoint family $\mathscr{F} \subseteq \mathcal{P}(\kappa)$ of cardinality $> \kappa$.)

Theorem 0.37 (Miller).

If $V = L$ then there is a Π^1_1 infinite mad family at ω.

However, motivated by some other considerations involving stronger hypotheses, Lücke and Müller asked:

Question 0.38 (Lücke, Müller).

Do sufficiently strong large cardinal properties of κ imply that there is no $\Pi_1(\mathcal{H}_\kappa \cup \{\kappa\})$ almost disjoint family of cardinality $> \kappa$?

Well:

Theorem 0.39 (S.).

It is consistent relative to large cardinals / mouse existence hypotheses that:

- *κ is a Woodin limit of Woodins and there is a $\Pi_1(\{\kappa\})$ mad family of cardinality $> \kappa$, or*
- *κ is superstrong and there is a $\Pi_1(\{A, \kappa\})$ mad family of cardinality $> \kappa$, for some $A \subseteq \omega_1$.*

Beyond Woodin limit of Woodins, need mouse existence hypotheses.

Independent families

Definition 0.40.

Let κ be an infinite cardinal. Recall an independent family (at κ) is a set

$$\mathscr{F} \subseteq \mathcal{P}(\kappa)$$

such that:

- A is unbounded in κ for every $A \in \mathscr{F}$, and
- for all finite sets $\mathscr{A}, \mathscr{B} \subseteq \mathscr{F}$ with $\mathscr{A} \cap \mathscr{B} = \emptyset$, we have

$$\left(\bigcap \mathscr{A}\right) \cap \left(\bigcap \mathscr{B}'\right) \neq \emptyset,$$

  where $\mathscr{B}' = \{\kappa \backslash B \mid B \in \mathscr{B}\}$.

And $\mathscr{F}$ maximal if there is no independent $\mathscr{F}'$ with $\mathscr{F} \subsetneq \mathscr{F}'$.

The relative consistency of $\Pi_1(X)$ mad families adapts directly to $\Pi_1(X)$ maximal independent families:

Theorem 0.41 (S.).

It is consistent relative to large cardinals / mouse existence hypotheses that:

- κ *is a Woodin limit of Woodins and there is a* $\Pi_1(\{\kappa\})$ *maximal independent family of cardinality* $> \kappa$, *or*
- κ *is superstrong and there is a* $\Pi_1(\{A, \kappa\})$ *maximal independent family of cardinality* $> \kappa$, *for some* $A \subseteq \omega_1$.

(The proof is the same structure, but the combinatorics are a little different.)

Question 0.42.

Suppose μ is measurable, $\kappa > \mu$ a cardinal.
Let S_κ be the class of all U-$[0, \kappa)$-stable sets. Suppose $\kappa \in S_\kappa$.

- Can there be / is there a $\Sigma_1(S_\kappa)$ ultrafilter on κ? Or ultrafilter base?
- What about $\Pi_1(S_\kappa)$?
- What about a filter with other nice properties?

Thank you!