Strong almost disjointness and complex analysis

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Part I

Wetzel families

Wetzel's problem

Recall that a function $f : \mathbb{C} \to \mathbb{C}$ is called *entire* when it is complex differentiable everywhere.[1]

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Question (Wetzel '61?)

Let $\mathcal{F}$ be a set of entire functions $f : \mathbb{C} \to \mathbb{C}$. If $\{f(z) : f \in \mathcal{F}\}$ is countable for every $z \in \mathbb{C}$, is $\mathcal{F}$ itself countable?[2]

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Theorem (Erdős '63)

The answer is yes iff CH *is false. In other words, the existence of (pairwise distinct)* $\langle f_\alpha : \alpha < \omega_1 \rangle$ *such that*

$$\forall z \in \mathbb{C}(|\{f_\alpha(z) : \alpha < \omega_1\}| < \omega_1)$$

is equivalent to CH.

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Lemma (Erdős)

For any countable dense $X \subseteq \mathbb{C}$ *and any countable* $Y \subseteq \mathbb{C}$ *there is a non-constant entire* f*, such that*

$$f[Y] \subseteq X.$$

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Fix X countable dense
$\langle z_\alpha : \alpha < \omega_1 \rangle \ldots$ enumeration of $\mathbb{C}$
$f_\alpha[\{z_\beta : \beta < \alpha\}] \subseteq X$ forms a Wetzel family

Erdős' problem

Question (Erdős '63)

In general, without assuming CH, is there a family $\mathcal{F}$ of size $2^{\aleph_0}$ such that at each $z \in \mathbb{C}$,

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> Unfortunately I am unable to decide the following question: Can one construct a family of distinct entire functions f_α $(1 \leq \alpha < \Omega_c)$ such that for every z the set $\{f_\alpha(z)\}$ has power less than $\mathfrak{c}$? We proved that the construction is possible if $\mathfrak{c} = \aleph_1$, but for $\mathfrak{c} > \aleph_1$ our proof breaks down. Paul Cohen's recent proof of the independence of the continuum hypothesis gives this problem some added interest.

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After a suggestion by Martin Goldstern:

Definition

We call a family $\mathcal{F}$ as above a *Wetzel family*.

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Erdős' problem is independent of $\mathrm{ZFC} + \neg\mathrm{CH}$*. More precisely, over a ground model satisfying* GCH*:*

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2. *There is a cardinal and cofinality preserving forcing extension in which* $2^{\aleph_0} = \aleph_{\omega_1}$ *and there is a Wetzel family.*

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Question (Kumar-Shelah)

Is a Wetzel family consistent with regular continuum, e.g. $2^{\aleph_0} = \aleph_2$?

Universal sets

An approach suggested by Kumar-Shelah is to make Erdős' orginial proof somehow work.

Definition

We call $X \subseteq \mathbb{C}$ universal (for entire functions) if $|X| < 2^{\aleph_0}$ and whenever $|Y| < 2^{\aleph_0}$, there is a non-constant entire

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Lemma

The existence of a universal set implies that $2^{\aleph_0}$ is a successor cardinal.

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Question

Is a universal set consistent with continuum $\aleph_3$?

Hooray!

Theorem (S.-Weinert)

(GCH) Let κ have uncountable cofinality. Then there is a cardinal and cofinality preserving forcing extension with $2^{\aleph_0} = \kappa$ and a Wetzel family. Moreover, if κ is regular, we can also add MA*.*

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Does MA, MA $+ 2^{\aleph_0} = \aleph_2$ or PFA imply that there is a Wetzel family?

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Conjecture: PFA works.

Part II

Strong almost disjointness

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This somewhat motivates Wetzel's original question.

Now suppose for instance that the continuum is λ^+. Then we can think of a Wetzel family as a family $\mathcal{F}$ of functions $f : \lambda^+ \to \lambda$ so that for all $f \neq g \in \mathcal{F}$, $f \cap g$ is countable.

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More generally, if $2^{\aleph_0} = \kappa$, a Wetzel family gives a "σ-almost disjoint" family $\mathcal{F} \subseteq \prod_{\alpha<\kappa} \mu_\alpha$ of size κ, where $\mu_\alpha < \kappa$ for all $\alpha < \kappa$.

Almost disjointness

So in order e.g. to answer whether a Wetzel family with continuum of size $\aleph_3$ is consistent, we must also answer:

Question

Is a σ-ad family in ${\omega_2}^{\omega_3}$ of size ω_3 consistent at all?

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Question

Is a σ-ad family in ${\omega_2}^{\omega_3}$ of size ω_3 consistent at all?

In the proof of our main theorem, we in fact use a preparatory forcing to first get a positive answer to this question and then force again to add the entire functions.

In the case of ω_2, the question has a positive answer in ZFC: There is a σ-ad family in ${\omega_1}^{\omega_2}$ of size ω_2.

Almost disjointness

A very similar question has been asked in the context of a question by Hajnal:

Question (Hajnal)

How long can chains in $(\omega_1^{\omega_1}, < /_{\mathsf{fin}})$ be?

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Zapletal noted the following:

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If there is a chain of length $\kappa + 1$ in $(\omega_1^{\omega_1}, < /_{\mathsf{fin}})$, then there is a strongly almost disjoint family[3] of size κ in ω^{ω_1}.

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Theorem (Zapletal '98)

Arbitrarily large strongly almost disjoint families in ω^{ω_1} are consistent. The same is true of ${\omega_n}^{\omega_{n+1}}$, for any $n \in \omega$.

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Question (Zapletal '98)

What about ${\omega_\omega}^{\omega_{\omega+1}}$?

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Hooray!

Theorem (S.-Weinert)

(GCH) Let κ have uncountable cofinality and for each $\alpha < \kappa$, let $\mu_\alpha = \max(|\alpha|, \aleph_0)$. Then there is a cardinal and cofinality preserving forcing extension with a strongly almost disjoint family of size κ in $\prod_{\alpha<\kappa} \mu_\alpha$ and $2^{\aleph_0} = \kappa$.

In particular, for any λ, arbitrarily large strongly almost disjoint families are consistent in λ^{λ^+}.

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(GCH) Let κ *have uncountable cofinality and for each* $\alpha < \kappa$*, let* $\mu_\alpha = \max(|\alpha|, \aleph_0)$*. Then there is a cardinal and cofinality preserving forcing extension with a strongly almost disjoint family of size* κ *in* $\prod_{\alpha<\kappa} \mu_\alpha$ *and* $2^{\aleph_0} = \kappa$*.*

In particular, for any λ, arbitrarily large strongly almost disjoint families are consistent in λ^{λ^+}.

Theorem (S.-Weinert, ?)

MA *implies that there is a strongly almost disjoint family of size* ω_2 *in* ${\omega_1}^{\omega_2}$*.*

Thinning out

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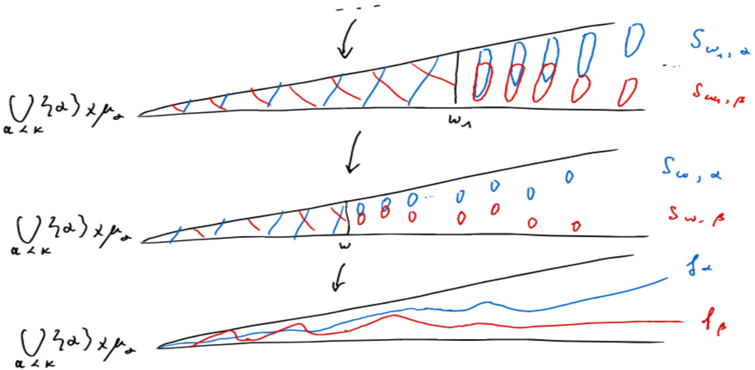
Suppose for instance that $\{A_\alpha : \alpha < \lambda\} \subseteq [\omega_1]^{\omega_1}$ is σ-ad. Then adding a strongly ad family $\{B_\alpha : \alpha < \lambda\}$ with $B_\alpha \subseteq A_\alpha$ with finite conditions is ccc. We are thinning out the A_α's.

For every regular $\lambda \leq \kappa$, $\langle S_{\lambda,\alpha} : \alpha < \kappa \rangle$ is λ^+-ad, all sections of $S_{\lambda,\alpha}$ after index λ have size λ.



Madness

The technique also leads to the following quite interesting result:

Theorem (S.)

Let κ be regular. Then it is consistent that there is a κ-mad family (of arbitrarily large size) that is strongly ad.
I.e. $\mathcal{A}$ such that for every $X \in [\kappa]^\kappa$ there is $A \in \mathcal{A}$, $|A \cap X| = \kappa$ and $\forall A_0 \neq A_1 \in \mathcal{A}$, $|A_0 \cap A_1| < \omega$.

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Lemma

Let $\mathcal{A} \subseteq [\omega_1]^{\omega_1}$ be an (ω_1-)mad family. Then there is a ccc forcing adding a refinement of $\mathcal{A}$ that is ω_1-mad and strongly almost disjoint.

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In fact

Theorem (S.)

MA $+ 2^{\aleph_0} = 2^{\aleph_1}$ *implies that every* ω_1*-mad family has a strongly ad refinement that is* ω_1*-mad.*

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Question

Is the conclusion consistent for $\kappa > \omega_1$?

Thank you!

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