The SLO principle for Borel subsets of the generalized Cantor space

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Joint work with Luca Motto Ros and Philipp Schlicht

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Generalized descriptive set theory

A cofinality-indepedent approach

The higher analogue of the classical setting, obtained by replacing ω with κ... or $\mathrm{cof}(\kappa)$!

Remark: Let κ be an infinite cardinal. Then $\kappa^{<\kappa} = \kappa$ is equivalent to $2^{<\kappa} = \kappa$ and κ *regular*.

Our setup

Let κ be an uncountable cardinal that satisfies the condition $2^{<\kappa} = \kappa$.

The generalized Cantor and Baire spaces

Let λ, μ be cardinals, with μ infinite and $\lambda \geq 2$.
We equip the set ${}^{\mu}\lambda = \{x \mid x : \mu \to \lambda\}$ with the *bounded topology* τ_b, generated by the sets

$$N_s({}^{\mu}\lambda) := \{\, x \in {}^{\mu}\lambda \mid s \subseteq x \,\}, \qquad s \in {}^{<\mu}\lambda.$$

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Theorem

If κ is a singular cardinal and $2^{<\kappa} = \kappa$, then ${}^{\kappa}2 \approx {}^{\mathrm{cof}(\kappa)}\kappa$.

Wadge Reductions

Definition

Given $A, B \subseteq {}^{\omega}2$, let

$$A \leq_{\mathrm{W}} B$$

if there exists a continuous $f : {}^{\omega}2 \to {}^{\omega}2$ such that $f^{-1}(B) = A$.

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if there exists a continuous $f : {}^{\omega}2 \to {}^{\omega}2$ such that $f^{-1}(B) = A$.

- Notice that $A \leq_{\mathrm{W}} B$ if and only if $\neg A \leq_{\mathrm{W}} \neg B$.
- Continuous reducibility is a transitive and reflexive relation, that is, a preorder.

Wadge Hierarchy

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We set:

- $A <_{\mathrm{W}} B$ iff $A \leq_{\mathrm{W}} B$ and $B \not\leq_{\mathrm{W}} A$.
- $A \equiv_{\mathrm{W}} B$ iff $A \leq_{\mathrm{W}} B$ and $B \leq_{\mathrm{W}} A$.

The equivalence classes induced by $\leq_{\mathrm{W}}$ are called Wadge degrees

$$[A]_{\mathrm{W}} = \{B \mid A \equiv_{\mathrm{W}} B\}$$

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Note that the preorder $\leq_{\mathrm{W}}$ induces a partial order on the Wadge degrees: We call this partial order the **Wadge hierarchy**.

Wadge Hierarchy

Theorem (Martin, Monk)

$\leq_W$ is well-founded on $\mathbf{Bor}({}^\omega 2)$.

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Wadge's Lemma

For all $A, B \in \mathbf{Bor}({}^{\omega}2)$,

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\omega}2 \setminus B \leq_{\mathsf{W}} A.$$

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The **Wadge Semi-Linear Ordering principle** ($\mathsf{SLO}^{\mathrm{W}}$) is the statement: For all sets $A, B \subseteq {}^{\omega}2$

$$A \leq_{\mathrm{W}} B \quad \text{or} \quad {}^{\omega}2 \setminus B \leq_{\mathrm{W}} A.$$

Given $\mathbf{\Gamma}$ boldface pointclass, we write $\mathsf{SLO}^{\mathrm{W}}(\mathbf{\Gamma})$ if $\mathsf{SLO}^{\mathrm{W}}$ holds for any $A, B \in \mathbf{\Gamma}$.

Wadge game

For any $A, B \subseteq {}^{\omega}2$, the Wadge game $G_{\mathrm{W}}(A, B)$ on ${}^{\omega}2$ is:

$$\begin{array}{c|cccccccc} \mathbf{I} & x_0 & & x_1 & & x_2 & & \cdots & \\ \hline \mathbf{II} & & y_0 & & \mathrm{P} & & y_1 & & \cdots \end{array}$$

Player **II** is allowed to "pass" at some stages.
Player **II** wins the game if $y \in {}^{\omega}2$ and $x \in A \iff y \in B$.

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Fact

- **II** has a winning strategy in $G_{\mathrm{W}}(A, B) \iff A \leq_{\mathrm{W}} B$.
- **I** has a winning strategy in $G_{\mathrm{W}}(A, B) \implies {}^{\omega}2 \setminus B \leq_{\mathrm{W}} A$.

Some consequences of $\mathsf{SLO}^{\mathsf{W}}$

- Antichains have size at most 2, and they are of the form $\{[A]_{\mathsf{W}}, [\neg A]_{\mathsf{W}}\}$ for some $A \subseteq {}^{\omega}2$ such that $A \not\leq_{\mathsf{W}} \neg A$.

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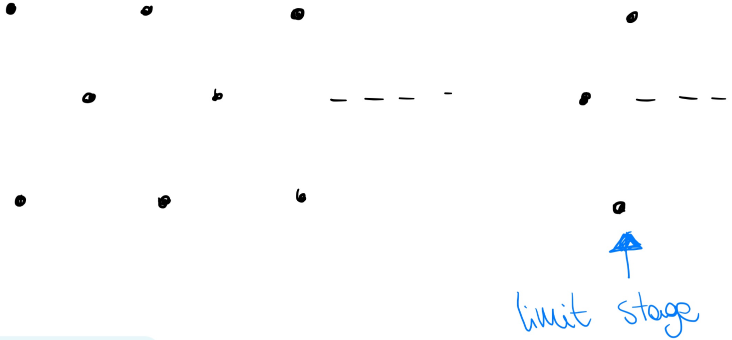

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Definition

Let $\boldsymbol{\Gamma}$ be a boldface pointclass.

- A set $A \subseteq {}^{\omega}2$ is $\boldsymbol{\Gamma}$-*hard* if for all $B \in \boldsymbol{\Gamma}\,({}^{\omega}2)$, $B \leq_{\mathrm{W}} A$.
- The set A is $\boldsymbol{\Gamma}$-*complete* if it is $\boldsymbol{\Gamma}$-hard and $A \in \boldsymbol{\Gamma}({}^{\omega}2)$.

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- Assume $\mathsf{SLO}^{\mathsf{W}}$ holds. Let $\mathbf{\Gamma}$ be a non selfdual boldface pointclass, then

$$A \text{ is } \mathbf{\Gamma}\text{-complete} \iff A \in \mathbf{\Gamma}({}^{\omega}2) \setminus \breve{\mathbf{\Gamma}}({}^{\omega}2).$$

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$\Rightarrow]$

Since A is Γ-hard, $\forall B \in \Gamma$ $B \leq_w A$.

If it were $A \in \check{\Gamma}$ $\Rightarrow$ $\Gamma = \check{\Gamma}$ ↯

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$[\Leftarrow$

Let $A \in \Gamma \setminus \check{\Gamma}$.

For any $B \in \Gamma$, by SLO^{w}: $\left\{ \begin{array}{l} B \leq_w A \\ \quad \text{or} \\ A \leq_w \underbrace{\neg B}_{\in \check{\Gamma}} \end{array} \right.$

Some consequences of $\mathsf{SLO}^{\mathsf{W}}$

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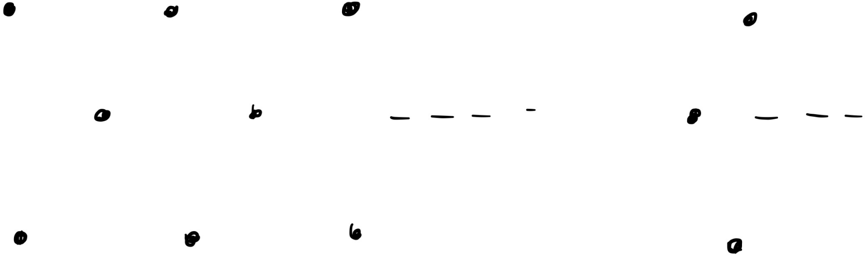
Wadge Hierarchy



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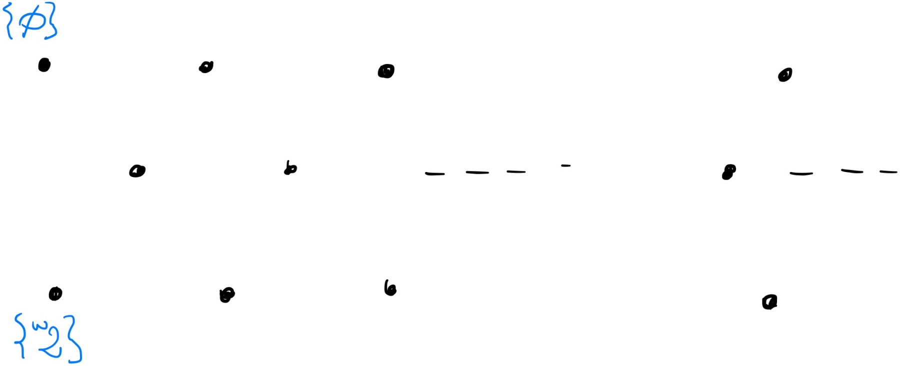

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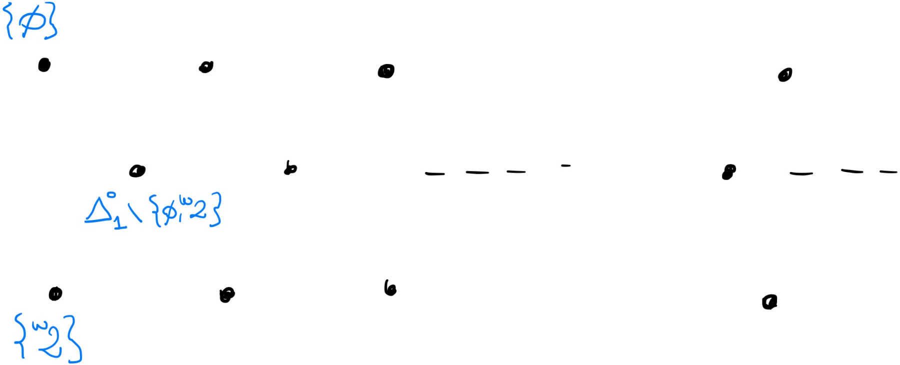

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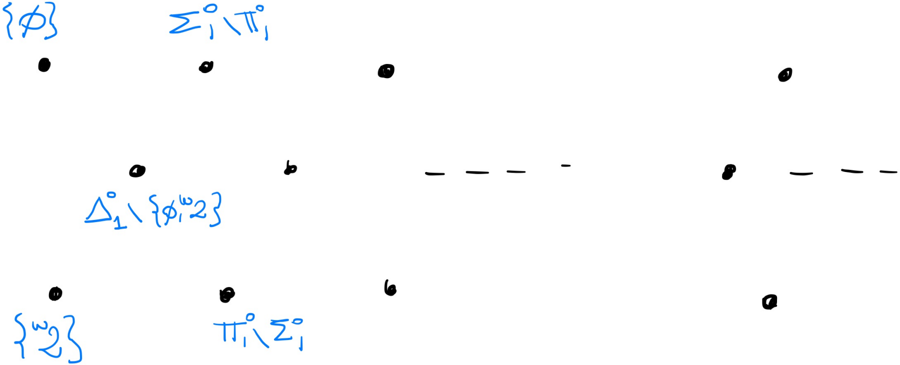

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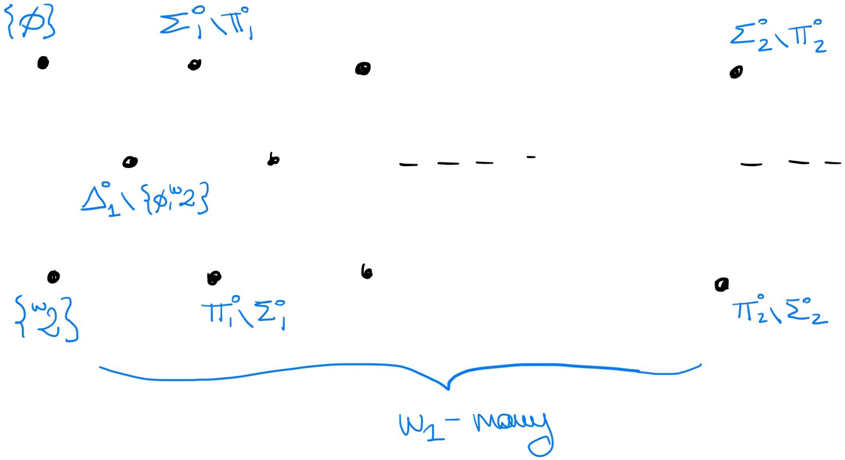

Difference Hierarchy - DTS

$\theta \in \mathrm{Ord}$ can be uniquely written as $\theta = \lambda + n$ with λ limit or 0 and $n < \omega$.

Definition

Let $\theta \geq 1$ be an ordinal. If $(C_\eta)_{\eta<\theta}$ is a decreasing sequence of subsets of a set X, we define $C = D_\theta\left((C_\eta)_{\eta<\theta}\right) \subseteq X$ by

$$x \in C \iff \begin{cases} x \in \bigcap_{\eta<\theta} C_\eta \vee \min\{\eta < \theta \mid x \notin C_\eta\} \text{ is odd} & \text{for } \theta \text{ odd} \\ x \notin \bigcap_{\eta<\theta} C_\eta \wedge \min\{\eta < \theta \mid x \notin C_\eta\} \text{ is odd} & \text{for } \theta \text{ even} \end{cases}$$

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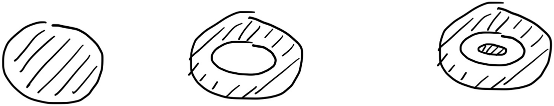
$D_1(C_0) = C_0$ $\quad$ $D_2(C_0, C_1) = C_0 \setminus C_1$ $\quad$ $D_3(C_0, C_1, C_2) = C_0 \setminus C_1 \cup C_2$ $\quad$ $D_\omega((C_i)_{i<\omega}) =$



$\bigcup_{i<\omega}(C_{2i} \setminus C_{2i+1})$

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Consider $X = ({}^\omega 2, \tau)$. For $1 \leq \theta < \omega_1$, we let

$$\mathbf{D}_\theta\left(\mathbf{\Pi}^0_1\right) = \left\{ D_\theta\left((C_\eta)_{\eta<\theta}\right) \mid C_\eta \in \mathbf{\Pi}^0_1 \text{ for every } \eta < \theta \right\}.$$

We also define $\check{\mathbf{D}}_\theta\left(\mathbf{\Pi}^0_1\right)$ to be the dual class of $\mathbf{D}_\theta\left(\mathbf{\Pi}^0_1\right)$.

Difference Hierarchy - DTS

Theorem (Hausdorff, Kuratowski)

In every polish space X and for any $1 \leq \alpha < \omega_1$,

$$\mathbf{\Delta}^0_{\alpha+1}(X) = \bigcup_{1 \leq \theta < \omega_1} \mathbf{D}_\theta \left(\mathbf{\Pi}^0_\alpha(X) \right)$$

Difference Hierarchy - GDST

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Consider $X = ({}^{\kappa}2, \tau_b)$. For $1 \leq \theta < \kappa^+$, we let

$$\mathbf{D}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right) = \left\{D_\theta\left((C_\eta)_{\eta<\theta}\right) \mid C_\eta \in \mathbf{\Pi}^0_1(\kappa^+) \text{ for every } \eta < \theta\right\}.$$

We also define $\check{\mathbf{D}}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right)$ to be the dual class of $\mathbf{D}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right)$.

A counterexample to Hausdorff-Kuratowski in GDST

Theorem

Let $X \subseteq {}^{\kappa}2$. If $Y \subseteq X$ is non-empty, dense and codense in $\overline{X}$, then $Y \notin \mathbf{D}_{\theta}(\mathbf{\Pi}^0_1(X, \kappa^+))$ for any $\theta < \kappa^+$.

Consider the sets

$$X := \{x \in {}^{\kappa}2 \mid |\{\alpha < \kappa \mid x(\alpha) = 1\}| < \aleph_0\}$$

and

$$Y := \{x \in {}^{\kappa}2 \mid \exists n < \omega |\{\alpha < \kappa \mid x(\alpha) = 1\}| = 2n\}.$$

Define also

$$Y^c := X \setminus Y = \{x \in X \mid \exists n < \omega |\{\alpha < \kappa \mid x(\alpha) = 1\}| = 2n + 1\}$$

SLO$^{\mathsf{W}}$ in GDST

Definition

Given $A, B \subseteq {}^{\kappa}2$, let

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if there exists a continuous $f : {}^{\kappa}2 \to {}^{\kappa}2$ such that $f^{-1}(B) = A$.

The *generalized Wadge Semi-Linear Ordering principle* ($\mathsf{SLO}^{\mathsf{W}}_{\kappa}$) says:
For all sets $A, B \subseteq {}^{\kappa}2$

$$A \leq_{\mathsf{W}} B \quad \text{or} \quad {}^{\kappa}2 \setminus B \leq_{\mathsf{W}} A.$$

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The *generalized Wadge Semi-Linear Ordering principle* ($\mathrm{SLO}^{\mathrm{W}}_{\kappa}$) says:
For all sets $A, B \subseteq {}^{\kappa}2$

$$A \leq_{\mathrm{W}} B \quad \text{or} \quad {}^{\kappa}2 \setminus B \leq_{\mathrm{W}} A.$$

However, there is no κ^+-Borel determinacy for $\kappa > \omega$...

Generalized Gale-Stewart game

Let κ, λ be cardinals, with κ infinite and $\lambda \geq 2$.
Given $A \subseteq {}^{\kappa}\lambda$, the generalized Gale-Stewart game $G^{\lambda}_{\kappa}(A)$ is

I	a_0		a_2		$\cdots$		a_ω		$\cdots$
II		a_1		a_3		$\cdots$		$a_{\omega+1}$	$\cdots$

Let $a := \langle a_0, a_1, ..., a_\omega, ... \rangle \in {}^{\kappa}\lambda$. Player **I** wins if $a \in A$ and **II** wins if $a \notin A$.

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Let $a := \langle a_0, a_1, ..., a_\omega, ... \rangle \in {}^{\kappa}\lambda$. Player **I** wins if $a \in A$ and **II** wins if $a \notin A$.

Fact

Let $\kappa > \omega$ and let $A \subseteq {}^{\omega}2$. Then, there is an extension $\overline{A} \subseteq {}^{\kappa}2$ of A such that $\overline{A} \in \boldsymbol{\Delta}_1^0(\kappa^+)$ and $G_{\kappa}^2(\overline{A})$ is equivalent to $G_{\omega}^2(A)$.

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Fact

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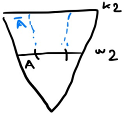
$\overline{A} = \bigcup\{N_s : \mathrm{lh}(s) = \omega \wedge s \in A\}$



SINGULAR CASE

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is singular

Theorem (Motto Ros, Schlicht)

(AC). Let X be an uncountable ultrametric Polish space. Then there is a map $\psi : \mathcal{P}(\omega) \to \mathcal{P}(X)$ such that for all $a, b \subseteq \omega$

1. if $a \subseteq b$, then $\psi(a) \leq_{\mathsf{L}} \psi(b)$;
2. if $\psi(a) \leq_{\mathbf{Bor}(X)} \psi(b)$, then $a \subseteq b$.

In particular, $(\mathcal{P}(\omega), \subseteq)$ embeds into the $\mathcal{F}$-hierarchy on X for every reducibility $\mathsf{L} \subseteq \mathcal{F} \subseteq \mathbf{Bor}(X)$.

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Theorem (Motto Ros, P., Schlicht)

Let $\kappa > \omega$ with $\mathrm{cof}(\kappa) = \omega$. Then, $(\mathcal{P}(\omega), \subseteq)$ embeds into the W-hierarchy on the $\boldsymbol{\Delta}^0_2(\kappa^+)$ subsets of ${}^{\omega}\kappa$.

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is singular

Theorem (Motto Ros, P., Schlicht)

Let μ be an uncountable cardinal s.t. $\mu^{<\mu} = \mu$. Then, there is a map $\psi : \mathcal{P}(\mu) \to \mathcal{P}({}^{\mu}\mu)$ such that for all $a, b \subseteq \mu$

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In particular, $(\mathcal{P}(\mu), \subseteq)$ embeds into the $\mathcal{F}$-hierarchy on ${}^{\mu}\mu$ for every reducibility $\mathsf{L} \subseteq \mathcal{F} \subseteq \mathsf{Bor}(\mu^+)$.

Theorem (Motto Ros, P., Schlicht)

Let κ be a singular cardinal. Then $(\mathcal{P}(\mathrm{cof}(\kappa)), \subseteq)$ embeds into the W-hierarchy on the $\mathbf{\Delta}^0_2(\kappa^+)$ subsets of ${}^{\mathrm{cof}(\kappa)}\kappa$.

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is singular

Theorem (Motto Ros, P., Schlicht)

Let μ be an uncountable cardinal s.t. $\mu^{<\mu} = \mu$. Then, there is a map $\psi : \mathcal{P}(\mu) \to \mathcal{P}({}^{\mu}\mu)$ such that for all $a, b \subseteq \mu$

1. if $a \subseteq b$, then $\psi(a) \leq_{\mathsf{L}} \psi(b)$;
2. if $\psi(a) \leq_{\mathsf{Bor}(\mu^+)} \psi(b)$, then $a \subseteq b$.

In particular, $(\mathcal{P}(\mu), \subseteq)$ embeds into the $\mathcal{F}$-hierarchy on ${}^{\mu}\mu$ for every reducibility $\mathsf{L} \subseteq \mathcal{F} \subseteq \mathsf{Bor}(\mu^+)$.

Theorem (Motto Ros, P., Schlicht)

Let κ be a singular cardinal. Then $(\mathcal{P}(\mathrm{cof}(\kappa)), \subseteq)$ embeds into the W-hierarchy on the $\mathbf{\Delta}^0_2(\kappa^+)$ subsets of ${}^{\mathrm{cof}(\kappa)}\kappa$.

REGULAR CASE

Definition

Let $\mathcal{T} \subseteq {}^{<\kappa}\lambda$ with $\lambda \in \{2, \kappa\}$.

- $\mathcal{T}$ is pruned if for every $s \in \mathcal{T}$ there exists $x \in [\mathcal{T}]$ such that $s \subseteq x$.
- $\mathcal{T}$ is $< \kappa$-closed if every increasing sequence in $\mathcal{T}$ of length $< \kappa$ has an upper bound in $\mathcal{T}$,
- A node $s \in \mathcal{T}$ is splitting if there are two incomparable $t, t' \in \mathcal{T}$ extending s. The tree $\mathcal{T}$ is splitting if every node $s \in \mathcal{T}$ is splitting.
- $\mathcal{T}$ is κ-perfect if it is $< \kappa$-closed and cofinally splitting, i.e. if for every $t \in \mathcal{T}$ there exists a splitting node $u \in \mathcal{T}$ with $t \subseteq u$.
- A subset Y of ${}^{\kappa}\lambda$ is κ-perfect if $Y = [\mathcal{T}]$ with $\mathcal{T}$ a κ-perfect tree.
- A subset A of ${}^{\kappa}\lambda$ has the perfect set property if $\mid A \mid \leq \kappa$ or A has a κ-perfect subset.

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume $\mathsf{V} = \mathsf{L}$. Then, there are proper $\mathbf{\Sigma}^0_2(\kappa^+)$-sets which are not $\mathbf{\Sigma}^0_2(\kappa^+)$-complete. Hence, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{\Sigma}^0_2(\kappa^+))$ fails.

SLO$^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume V = L. Then, there are proper $\mathbf{\Sigma}^0_2(\kappa^+)$-sets which are not $\mathbf{\Sigma}^0_2(\kappa^+)$-complete. Hence, SLO$^{\mathsf{W}}_{\kappa}(\mathbf{\Sigma}^0_2(\kappa^+))$ fails.

Proof:

Let $T \subseteq {}^{<\kappa}2$ s.t.
- $[T]$ is κ-Baire
- $[T]$ has no isolated pts
- T has no κ-perfect subtree

Let $S \subseteq {}^{<\kappa}2$ be a κ-perfect tree

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Lücke, Motto Ros, Schlicht)

Assume V = L. If κ is an uncountable regular cardinal, then there is a closed subset of ${}^{\kappa}\kappa$ that does not satisfy the Hurewicz dichotomy.

Proposition (Lücke, Motto Ros, Schlicht)

Let $\mathcal{T} \subseteq {}^{<\kappa}\kappa$ be a pruned subtree with the following three properties:

1. $\mathcal{T}$ does not contain a perfect subtree;
2. the closed set $[\mathcal{T}]$ is κ-Baire,
3. every node in $\mathcal{T}$ is κ-splitting.

Then the closed set $[\mathcal{T}]$ does not satisfy the Hurewicz dichotomy.

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume V = L. Then, there are proper $\mathbf{\Sigma}^0_2(\kappa^+)$-sets which are not $\mathbf{\Sigma}^0_2(\kappa^+)$-complete. Hence, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{\Sigma}^0_2(\kappa^+))$ fails.

Proof:

Let $T \subseteq {}^{<\kappa}2$ s.t.
- $[T]$ is κ-Baire
- $[T]$ has no isolated pts
- T has no κ-perfect subtree

Let $S \subseteq {}^{<\kappa}2$ be a κ-perfect tree

$\forall s \in S$ pick $x_s \in [S]$ s.t. $s \subseteq x_s$ $\rightsquigarrow A = \{x_s : s \in S\}$

$\forall t \in T$ pick $x_t \in [T]$ s.t. $t \subseteq x_t$ $\rightsquigarrow B = \{x_t : t \in T\}$

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume V = L. Then, there are proper $\mathbf{\Sigma}^0_2(\kappa^+)$-sets which are not $\mathbf{\Sigma}^0_2(\kappa^+)$-complete. Hence, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{\Sigma}^0_2(\kappa^+))$ fails.

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$\forall t \in T$ pick $x_t \in [T]$ s.t. $t \subseteq x_t$ $\rightsquigarrow B = \{x_t : t \in T\}$

Note that: $A, B \in \Sigma^0_2(\kappa^+)$

A, B dense and codense in resp. $[S], [T]$.

$\mathsf{SLO}_\kappa^{\mathsf{W}}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume $\mathsf{V} = \mathsf{L}$. Then, there are proper $\boldsymbol{\Sigma}_2^0(\kappa^+)$-sets which are not $\boldsymbol{\Sigma}_2^0(\kappa^+)$-complete. Hence, $\mathsf{SLO}_\kappa^{\mathsf{W}}(\boldsymbol{\Sigma}_2^0(\kappa^+))$ fails.

Proof:

Fact

Let C be a closed set, $|C| > \kappa$, such that C is κ-Baire and there is $B \subseteq C$, $A \in \boldsymbol{\Sigma}_2^0(\kappa^+)$ dense and codense in C. Then, B is a proper $\boldsymbol{\Sigma}_2^0(\kappa^+)$-set.

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume $V = L$. Then, there are proper $\boldsymbol{\Sigma}^0_2(\kappa^+)$-sets which are not $\boldsymbol{\Sigma}^0_2(\kappa^+)$-complete. Hence, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Sigma}^0_2(\kappa^+))$ fails.

Proof:

CLAIM: $A \not\leq_w B$.

T.C., let $f : {}^{\kappa}2 \to {}^{\kappa}2$ cont. w/ $A = f^{-1}(B)$.

By induction on $s \in {}^{<\kappa}2$, we construct: $\langle u_s : s \in {}^{<\kappa}2 \rangle$ in S

$\langle v_s : s \in {}^{<\kappa}2 \rangle$ in T

st. $f[N_{u_s}] \subseteq N_{v_s}$.

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume V = L. Then, there are proper $\mathbf{\Sigma}^0_2(\kappa^+)$-sets which are not $\mathbf{\Sigma}^0_2(\kappa^+)$-complete. Hence, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{\Sigma}^0_2(\kappa^+))$ fails.

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By induction on $s \in {}^{<\kappa}2$, we construct: $\langle u_s : s \in {}^{<\kappa}2 \rangle$ in S

$\langle v_s : s \in {}^{<\kappa}2 \rangle$ in T

st. $f[N_{u_s}] \subseteq N_{v_s}$.

- $u_\emptyset = v_\emptyset = \emptyset$
- Given u_s, v_s: pick $x_0, x_1 \in N_{u_s} \cap [S]$ w/ $f(x_0) \neq f(x_1)$,
  let $v_{s^\frown 0}, v_{s^\frown 1} \supseteq v_s$ st. $v_{s^\frown i} \subseteq f(x_i)$ and $v_{s^\frown 0} \perp v_{s^\frown 1}$.
  By cont. of f, $\exists\, u_{s^\frown 0}, u_{s^\frown 1} \supseteq u_s$ st. $u_{s^\frown i} \subseteq x_i$, $u_{s^\frown 0} \perp u_{s^\frown 1}$ and $f(N_{u_{s^\frown i}}) \subseteq N_{v_{s^\frown i}}$.
- s of length α limit: $u_s = \bigcup_{\beta<\alpha} u_{s \restriction \beta}$, $v_s = \bigcup_{\beta<\alpha} v_{s \restriction \beta}$.

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Andretta)

$\mathsf{SLO}^{\mathsf{W}} \Longrightarrow \mathsf{PSP}$.

Theorem (Motto Ros, P., Schlicht)

Assume that $\mathsf{PSP}_{\kappa}(\boldsymbol{\Pi}^0_1(\kappa^+))$. Then, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ implies PSP_{κ}.

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume that $\mathsf{PSP}_{\kappa}(\boldsymbol{\Pi}^0_1(\kappa^+))$. Then, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ implies PSP_{κ}.

Proof: Let $X \subseteq {}^{\kappa}2$ and

$$\mathrm{G} = \{x \in {}^{\kappa}2 \mid \forall\alpha < \kappa\exists\beta > \alpha(x(\beta) = 0)\}$$

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

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$$\text{By } \mathsf{SLO}^{\mathsf{W}}_{\kappa} \quad \begin{cases} X \leq {}^{\kappa}2 \setminus G & (1) \\ \quad \text{or} & \\ G \leq X & (2) \end{cases}$$

$\mathrm{SLO}^{\mathrm{W}}_{\kappa}$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume that $\mathrm{PSP}_\kappa(\mathbf{\Pi}^0_1(\kappa^+))$. Then, $\mathrm{SLO}^{\mathrm{W}}_\kappa$ implies PSP_κ.

Proof: Let $X \subseteq {}^\kappa 2$ and

$$\mathrm{G} = \{x \in {}^\kappa 2 \mid \forall \alpha < \kappa \exists \beta > \alpha (x(\beta) = 0)\}$$

By SLO^w_κ

$$\begin{cases} X \le {}^\kappa 2 \setminus G & (1) \\ \text{or} & \\ G \le X & (2) \end{cases}$$

If (1) then $X = \bigcup_{\alpha<\kappa} C_\alpha$ w/ C_α closed $\forall \alpha < \kappa$.

Either $\forall \alpha \; |C_\alpha| \le \kappa \Rightarrow |X| \le \kappa$,

or $\exists \alpha \; |C_\alpha| > \kappa \xRightarrow[\mathrm{PSP}_\kappa(\Pi^0_1(\kappa^+))]{}$ C_α contains a κ-perfect set, hence X does.

$\mathsf{SLO}^{\mathsf{W}}_\kappa$ when κ is regular

Theorem (Motto Ros, P., Schlicht)

Assume that $\mathsf{PSP}_\kappa(\mathbf{\Pi}^0_1(\kappa^+))$. Then, $\mathsf{SLO}^{\mathsf{W}}_\kappa$ implies PSP_κ.

Proof: Let $X \subseteq {}^\kappa 2$ and

$$\mathrm{G} = \{x \in {}^\kappa 2 \mid \forall \alpha < \kappa \exists \beta > \alpha (x(\beta) = 0)\}$$

By SLO^w_κ $\begin{cases} X \leq {}^\kappa 2 \setminus G & (1) \\ \quad\text{or} & \\ G \leq X & (2) \end{cases}$

If (2) then $\exists f : {}^\kappa 2 \to {}^\kappa 2$ cont. s.t. $G = f^{-1}(X)$.

we can construct a κ-perfect tree $T \subseteq {}^{<\kappa}2$, $[T] \subseteq G$ s.t.

$f \restriction [T]$ injective $\Rightarrow$ $f([T])$ is a κ-perfect subset of X

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ when κ is regular

Let $\mathrm{G} = \{x \in {}^{\kappa}2 \mid \forall\alpha < \kappa\exists\beta > \alpha(x(\beta) = 0)\}$.

Theorem (Schlicht, Sziraki)

After a Levy-collapse of an inaccessible to κ^+, the following analogue of the *Kechris-Louveau-Woodin dichotomy* holds for all disjoint definable subsets $X, Y \subseteq {}^{\kappa}\kappa$:
Either there is a $\mathbf{\Sigma}^0_2(\kappa^+)$ set A separating X from Y, i.e. $X \subseteq A$ and $Y \cap A = \emptyset$ or there is a homeomorphism f from ${}^{\kappa}2$ onto a closed subset of ${}^{\kappa}\kappa$ such that $f(\mathrm{G}) \subseteq X$ and $f({}^{\kappa}2 \setminus \mathrm{G}) \subseteq Y$.

It is consistent that every proper $\mathbf{\Sigma}^0_2(\kappa^+)$-set is $\mathbf{\Sigma}^0_2(\kappa^+)$-complete.

$\mathrm{SLO}^{\mathrm{W}}_{\kappa}$ when κ is regular

Let $\mathrm{G} = \{x \in {}^{\kappa}2 \mid \forall \alpha < \kappa \exists \beta > \alpha (x(\beta) = 0)\}$.

Theorem (Schlicht, Sziraki)

After a Levy-collapse of an inaccessible to κ^+, the following analogue of the *Kechris-Louveau-Woodin dichotomy* holds for all disjoint definable subsets $X, Y \subseteq {}^{\kappa}\kappa$:
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It is consistent that every proper $\mathbf{\Sigma}^0_2(\kappa^+)$-set is $\mathbf{\Sigma}^0_2(\kappa^+)$-complete.

Question

Is it consistent that $\mathrm{SLO}^{\mathrm{W}}_{\kappa}(\mathbf{\Sigma}^0_2(\kappa^+))$ holds?

How far $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ holds

Fact 1

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Delta}^0_1(\kappa^+))$ holds.

let $A, B \in \Delta^0_1(\kappa^+)$. Assume $A, B \notin \{\emptyset, {}^{\kappa}2\} \Rightarrow \exists\, b \in B$

$\exists\, c \in {}^{\kappa}2 \setminus B$

Set $f : {}^{\kappa}2 \to {}^{\kappa}2$

$$x \longmapsto \begin{cases} b & \text{if } x \in A \\ c & \text{otherwise} \end{cases}$$

How far $\mathrm{SLO}^{\mathrm{W}}_{\kappa}$ holds

Fact 1

$\mathrm{SLO}^{\mathrm{W}}_{\kappa}(\mathbf{\Delta}^0_1(\kappa^+))$ holds.

Fact 2

Let $C \subseteq {}^{\kappa}2$. If $C \in \mathbf{\Pi}^0_1(\kappa^+) \setminus \mathbf{\Sigma}^0_1(\kappa^+)$, then C is $\mathbf{\Pi}^0_1(\kappa^+)$-complete.

Let $C \subseteq {}^{\kappa}2$, $C \in \Pi^0_1(\kappa^+) \setminus \Sigma^0_1(\kappa^+)$ and $D \in \Pi^0_1(\kappa^+)$. Let $x \in \partial C$.
We want $D \leq_W C$. Consider $G_W(D,C)$:

- As long as I plays nodes of T_D,
  II plays initial segments of x.
- If I ever reaches $t \in \partial T_D$,
  II picks an extension y of its previous play s.t. $y \notin C$
  and plays initial segments of y.

How far $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ holds

Fact 1

$\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Delta}^0_1(\kappa^+))$ holds.

Fact 2

Let $C \subseteq {}^{\kappa}2$. If $C \in \boldsymbol{\Pi}^0_1(\kappa^+) \setminus \boldsymbol{\Sigma}^0_1(\kappa^+)$, then C is $\boldsymbol{\Pi}^0_1(\kappa^+)$-complete.

Fact 3

Let $\boldsymbol{\Gamma}$ be a non selfdual boldface pointclass. If:

1. $\mathsf{SLO}^{\mathsf{W}}(\boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}})$ holds
2. A is $\boldsymbol{\Gamma}$-complete $\iff A \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$

then, $\mathsf{SLO}^{\mathsf{W}}(\boldsymbol{\Gamma})$ holds.

Hence, $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Sigma}^0_1(\kappa^+))$ and $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Pi}^0_1(\kappa^+))$ hold.

How far $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ holds

Structure of our work for $\theta > 1$:

1. Show that $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Gamma})$ holds for $\boldsymbol{\Gamma} = \mathbf{D}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+)) \cap \check{\mathbf{D}}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+))$.
2. Every proper $\mathbf{D}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+))$-subset $C \subseteq {}^\kappa 2$ is $\mathbf{D}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+))$-complete.
3. $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{D}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+)))$ holds.

How far $\mathsf{SLO}^{\mathsf{W}}_{\kappa}$ holds

Structure of our work for $\theta > 1$:

1. Show that $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\boldsymbol{\Gamma})$ holds for $\boldsymbol{\Gamma} = \mathbf{D}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+)) \cap \check{\mathbf{D}}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+))$.
2. Every proper $\mathbf{D}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+))$-subset $C \subseteq {}^{\kappa}2$ is $\mathbf{D}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+))$-complete.
3. $\mathsf{SLO}^{\mathsf{W}}_{\kappa}(\mathbf{D}_{\theta}(\boldsymbol{\Pi}^0_1(\kappa^+)))$ holds.

Thank you!

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