Uniformization results and Feldman-Moore Theorem in generalized descriptive set theory

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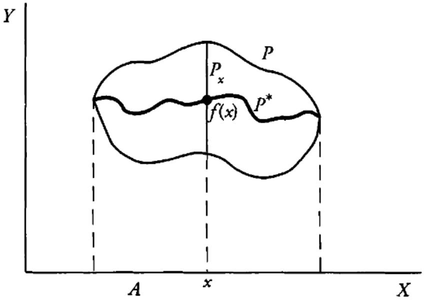
Workshop "Generalised Baire Space and Large Cardinals"
Bristol, February 8–10, 2024

Uniformization theorems

Given $P \subseteq X \times Y$, a **uniformization** of P is a subset $P^* \subseteq P$ such that for all $x \in X$

$$\exists y\, P(x,y) \iff \exists! y\, P^*(x,y).$$

Equivalently, P^* is the graph of a function f (called **uniformizing function**) with domain $\mathrm{proj}_X(P)$ such that $f(x) \in P_x$ for every $x \in A$.



Borel uniformizations

Fact

Let X, Y be standard Borel spaces. A set $P \subseteq X \times Y$ has a Borel uniformization if and only if $\mathrm{proj}_X(P)$ is Borel and there is a Borel uniformizing function $f\colon \mathrm{proj}_X(P) \to Y$ for P.

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General theme

Suppose that X, Y are Polish or standard Borel spaces, and that $P \subseteq X \times Y$ is Borel. Under which conditions there is a Borel uniformization of P?

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Suppose that X, Y are Polish or standard Borel spaces, and that $P \subseteq X \times Y$ is Borel. Under which conditions there is a Borel uniformization of P?

Today we are interested in "small section" uniformization results: if all the vertical sections of P are sufficiently small, then there is a Borel uniformization of P.

"Small section" uniformization results

Theorem (Lusin-Novikov)

Let X, Y be standard Borel spaces, and $P \subseteq X \times Y$ a Borel set with countable vertical sections. Then $P = \bigcup_{n \in \omega} P_n$ with each P_n a Borel set with vertical sections of size 1.

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Theorem (???)

Let X be a standard Borel space, Y a Polish space, and $P \subseteq X \times Y$ a Borel set with compact vertical sections P_x. Then the map $x \mapsto P_x$ from X to $K(Y)$ (endowed with the Vietoris topology) is Borel.

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Theorem (Arsenin-Kunugui)

Let X be a standard Borel space, Y a Polish space, and $P \subseteq X \times Y$ a Borel set whose vertical sections P_x are σ-compact (= countable unions of compact sets). Then P has a Borel uniformization.

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The importance of this theorem in descriptive set theory cannot be underestimated!

[Study of CBERs, Borel combinatorics, definable paradoxical decompositions, ...]

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- There are CBERs which are not hyperfinite.

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Weiss' conjecture

Any Borel action of a countable amenable group on a standard Borel space induces a hyperfinite CBER.

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Can we have "small section" uniformization results and/or an analogue of the Feldman-Moore theorem in GDST?

Interludio: the regular case

Assume that $\kappa > \omega$ is regular.

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Theorem (S.-D. Friedman-Hyttinen-Kulikov)

If E is an orbit equivalence relation on ${}^\kappa 2$ induced by a κ^+-Borel action of a (discrete) group of size at most κ, then $E \leq^\kappa_B E^\kappa_0$, where E^κ_0 is defined on 2^κ by $x \; E^\kappa_0 \; y \iff \exists \alpha < \kappa \, \forall \beta \geq \alpha \, (x(\beta) = y(\beta))$.

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Moreover, since κ is uncountable every orbit equivalence relation induced by a $\leq \kappa$-sized discrete group is **hyper-κ-small**, i.e. it can be written as an increasing union of size κ of κ^+-Borel equivalence relation which are κ-small (= all their classes have size $< \kappa$).

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Moreover, since κ is uncountable every orbit equivalence relation induced by a $\leq \kappa$-sized discrete group is **hyper-κ-small**, i.e. it can be written as an increasing union of size κ of κ^+-Borel equivalence relation which are κ-small ($=$ all their classes have size $< \kappa$). However:

Theorem (S.-D. Friedman-Hyttinen-Kulikov)

Assume $\mathsf{V} = \mathsf{L}$. Then there is a κ^+-Borel equivalence relation E whose classes have size 2 which is not induced by a κ^+-Borel action of a (discrete) group of size $\leq \kappa$.

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2. Is there a κ^+-Borel equivalence relation on ${}^{\kappa}2$ with classes of size at most κ which is not κ^+-Borel reducible to E_0^{κ}?
3. Can we have (at least consistently) "small section" uniformization results for κ^+-Borel subsets of ${}^{\kappa}2$?

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Recall that in GDST we must replace ω with either λ or $\mathrm{cof}(\lambda)$, so "countable" should be translated to "of size $\leq \lambda$" or remain "of size $\leq \omega$". Similarly, "compact" could be replaced by "λ-Lindelöf" or stay the same.

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We first consider the second option and look at λ-Borel sets with countable vertical sections, or with compact vertical sections.

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3. There is a sequence $(\zeta_n^P)_{n\in\omega}$ of λ-Borel functions $\zeta_n^P \colon \mathrm{proj}_X(P) \to Y$ such that for all $x \in \mathrm{proj}_X(P)$ the set $\{\zeta_n^P(x) \mid n \in \omega\}$ is dense in P_x.

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4. There is a sequence $(\varrho^P_n)_{n\in\omega}$ of λ-Borel functions $\varrho^P_n \colon \mathrm{proj}_X(P) \to Y$ such that $P_x = \{\varrho^P_n \mid n \in \omega\}$ for all $x \in \mathrm{proj}_X(P)$.

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5. The set P can be written as $P = \bigcup_{n\in\omega} P_n$ where the sets P_n are pairwise disjoint λ-Borel sets with vertical sections of size 1.

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Proposition

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Let $P \subseteq X \times Y$ be λ-Borel and with countable vertical sections. Pick a closed set $F \subseteq X \times {}^{\omega}\lambda$ and a λ-Borel isomorphism $f \colon F \to P$ such that $\mathrm{proj}_X(w) = \mathrm{proj}_X(f(w))$ for all $w \in F$.

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Countable vertical sections

Proposition

Let E be an equivalence relation on a standard λ-Borel space X that can be written as $E = \bigcup_{\alpha<\mu} P_\mu$ with $\omega \leq \mu \leq \lambda$ and each P_μ a λ-Borel set with vertical sections of size 1. Then there is a (discrete) group G of size $\leq \mu$ acting on X by λ-Borel isomorphisms (in fact, involutions) which generates E. If moreover $\mu > \omega$, then E is hyper-μ-small.

Countable vertical sections

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Let E be an equivalence relation on a standard λ-Borel space X that can be written as $E = \bigcup_{\alpha<\mu} P_\mu$ with $\omega \leq \mu \leq \lambda$ and each P_μ a λ-Borel set with vertical sections of size 1. Then there is a (discrete) group G of size $\leq \mu$ acting on X by λ-Borel isomorphisms (in fact, involutions) which generates E. If moreover $\mu > \omega$, then E is hyper-μ-small.

Generalized Feldman-Moore Theorem

Let X be a standard λ-Borel space. Then E is a countable λ-Borel equivalence relation on X if and only if it is the orbit equivalence relation induced by a λ-Borel action of a countable (discrete) group G on X.

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1. There is a λ-Borel uniformization of P.
2. The map from $\mathrm{proj}_X(P)$ to $K(Y) \subseteq F(Y)$ sending $x \in \mathrm{proj}_X(P)$ to P_x is λ-Borel.

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3. There is a sequence $(\zeta_n^P)_{n \in \omega}$ of λ-Borel functions $\zeta_n^P \colon \mathrm{proj}_X(P) \to Y$ such that for all $x \in \mathrm{proj}_X(P)$ the set $\{\zeta_n^P(x) \mid n \in \omega\}$ is dense in P_x.

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4. There is a sequence $(\varrho_\alpha^P)_{\alpha < 2^{\aleph_0}}$ of λ-Borel maps $\varrho_\alpha^P \colon \mathrm{proj}_X(P) \to Y$ such that $P_x = \{\varrho_\alpha^P \mid \alpha < 2^{\aleph_0}\}$ for all $x \in \mathrm{proj}_X(P)$.

[Recall that $\omega < 2^{\aleph_0} < \lambda$ by choice of λ.]

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   [Recall that $\omega < 2^{\aleph_0} < \lambda$ by choice of λ.]
5. The set P can be written as $P = \bigcup_{\alpha<2^{\aleph_0}} P_\alpha$ where the sets P_α are pairwise disjoint λ-Borel sets with vertical sections of size 1.

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Moreover, E is hyper-$2^{\aleph_0}$-small.

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Hence the more general case is given by 2 (which is equivalent to 4 and 5). This is still work in progress...

...more on the blackboard!!

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Thank you for your attention!