Large Cardinal Compactness

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Compactness

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Given any first order theory T, if every finite set of sentences from T is consistent (i.e., has a model), then T itself is consistent.

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- $<$ is a linear ordering, (first order statement)
- $x_i > x_{i+1}$ for $i < \omega$,
- $<$ is a well-ordering: $\forall A \exists x \in A \forall y \in A\ x \leq y$.

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κ is a strong compactness cardinal for $\mathcal{L}^2$ if and only if there is an extendible cardinal $\nu \leq \kappa$. In particular, the least strong compactness cardinal for $\mathcal{L}^2$ is the least extendible cardinal.

A cardinal ν is *extendible* if $\forall \eta > \nu \, \exists \zeta \, \exists j \colon V_\eta \to V_\zeta \; \mathrm{crit}(j) = \nu$ and $j(\nu) > \eta$.

Strongly compact cardinals and more

By their very definition, a cardinal κ is *strongly compact* if κ is a strong compactness cardinal for the logic $\mathcal{L}_{\kappa,\kappa}$ that is first order logic together with infinitary conjunctions and disjunctions of size less than κ and simultaneous quantification over any number of less than κ many variables.

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Weak compactness and *measurability* can also be characterized by compactness properties of $\mathcal{L}_{\kappa,\kappa}$, considering only theories of size κ.

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Theorem (Makowsky, 1985)

Every abstract logic has a compactness cardinal if and only if Vopěnka's principle holds.

Vopěnka's principle is the statement that for any class of structures in a given signature, there's an elementary embedding between two of them.

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Let's first see what we can do with second order logic!

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Given a certain large cardinal property φ, let's try to find a sequence of $<\kappa$-consistent second order theories T_κ for cardinals κ so that T_κ is consistent if and only if some $\lambda \leq \kappa$ satisfies $\varphi(\lambda)$.

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On the other hand, the ultrapower embedding obtained from the measurability of some $\nu \leq \kappa$ easily yields the consistency of T_κ.

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An analogous theory for strong cardinals: Fix some cardinal $\lambda > \kappa$. Our language will be constant symbols c_x for $x \in V_{\kappa+1}$ and constant symbols d_γ for $\gamma < \lambda$. T_κ^λ contains:

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Supercompact cardinals

A similar approach also works for supercompact cardinals.

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For every cardinal κ, there is a certain (definable in κ) natural and rich class C of second order theories such that every theory in C is consistent if and only if there is a measurable cardinal that is $\leq\kappa$.

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We can't take C to be the class of all $<\kappa$-consistent theories, for this would give us an extendible cardinal by Magidor's result. We could take $C = \{T_\kappa\}$, but that would not be a very natural class of theories, and it would certainly not be rich (in the sense of containing as many theories as possible).

Outward Compactness - Basic Idea

Reminder: If $\varphi(\lambda) \equiv$ "λ is measurable", T_κ contains:

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But: It almost is.

The main concept

Let ZFC^* denote the fragment of ZFC with the axioms of separation and replacement for Σ_2-formulae only.

Definition 1

An $\mathcal{L}^2$-theory T is *$<\kappa$-outward consistent* if for all $\theta > \kappa$ with $T \in V_\theta$, the partial order $\mathrm{Col}(\omega, V_\theta)$ forces that whenever $N \models \mathrm{ZFC}^*$ is an outer model of V_θ^V which preserves κ as a cardinal, T is $<\kappa$-consistent in N.

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Definition 2

A cardinal κ is an *outward compactness cardinal* for $\mathcal{L}^2$ if all $<\kappa$-outward consistent theories are consistent.

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κ is an outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a measurable cardinal $\nu \leq \kappa$. In particular, the least measurable cardinal is the least outward compactness cardinal for $\mathcal{L}^2$.

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An $\mathcal{L}^2$-theory T is *weakly* $<\kappa$*-outward consistent* if for all $\theta > \kappa$ with $T \in V_\theta$ and all infinite cardinals $\lambda < \kappa$, the partial order $\mathrm{Col}(\omega, V_\theta)$ forces that whenever $N \models \mathrm{ZFC}^*$ is an outer model of V_θ^V with $V_\lambda^N = V_\lambda^V$ which preserves κ as a cardinal, T is $<\kappa$-consistent in N.

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κ is a strong outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a strong cardinal $\nu \leq \kappa$.

Strong cardinals

Definition 3

An $\mathcal{L}^2$-theory T is *weakly* $<\kappa$*-outward consistent* if for all $\theta > \kappa$ with $T \in V_\theta$ and all infinite cardinals $\lambda < \kappa$, the partial order $\mathrm{Col}(\omega, V_\theta)$ forces that whenever $N \models \mathrm{ZFC}^*$ is an outer model of V_θ^V with $V_\lambda^N = V_\lambda^V$ which preserves κ as a cardinal, T is $<\kappa$-consistent in N.

Definition 4

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κ is a strong outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a strong cardinal $\nu \leq \kappa$. In particular, the least strong cardinal is the least outward compactness cardinal for $\mathcal{L}^2$.

Further results

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- We also characterize when Ord is Woodin by a compactness property of abstract logics.
- Similarly for Vopěnka's principle, but as there's Makowsky's result, this is perhaps not as interesting.

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Let $\mathfrak{r} \subseteq \omega$ be a real that codes $\langle V_\theta^M, \in\rangle$ in a $\mathrm{Col}(\omega, V_\theta^M)$-generic extension of M.

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