LST numbers for regularity quantifiers

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LST Numbers

Definition

Let $Q_1, \ldots, Q_n$ be (Lindström) quantifiers. The *Löwenheim-Skolem-Tarski* number $\mathsf{LST}(Q_1, \ldots, Q_n)$ is the least κ such that for all first order $\mathcal{L}$ with $|\mathcal{L}| < \kappa$, every $\mathcal{L}$ structure $\mathcal{A}$ contains an $\mathcal{L} \cup \{Q_1, \ldots, Q_n\}$ elementary substructure $\mathcal{B} \prec \mathcal{A}$ with $|\mathcal{B}| < \kappa$.
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Example

$\mathrm{LST}() = \aleph_1$

Two standard quantifiers

Definition (The Härtig Quantifier)

$I(\varphi(x), \psi(y))$ is true in $\mathcal{A}$ if

$$|\{x : \mathcal{A} \vDash \varphi(x)\}|^V = |\{y : \mathcal{A} \vDash \psi(y)\}|^V$$

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Definition (The Equal Cofinality Quantifier)

$Q^{\text{e.c.}}(\varphi(x_1, x_2), \psi(y_1, y_2))$ is true in $\mathcal{A}$ if the two sets

$$\{(x_1, x_2) : \mathcal{A} \models \varphi(x_1, x_2)\}$$

and

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Assume GCH.

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Definition

$Q^\alpha(\varphi(x_1, x_2), \psi(y_1, y_2), \chi(z_1, z_2))$ is true in $\mathcal{A}$ if the sets X and Y and Z defined over $\mathcal{A}$ by φ, ψ and χ satisfy:

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- Z is a well order of order type less than α; and
- The equal cofinality of X and Y is in $\text{Reg}_{\text{o.t.}(Z)}$

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Theorem (H.T.)

If $\beta \leq \alpha$ *then* $\mathrm{LST}(I, Q^{\beta}) \leq \mathrm{LST}(I, Q^{\alpha})$ *unless* $\beta > \mathrm{LST}(I, Q^{\alpha})$.

An Upper Bound

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Theorem (H.T.)

Suppose that there are no Mahlo cardinals below α*. Then* $\mathrm{LST}(I, Q^{\alpha}) \geq \min(\mathrm{Reg}_{\alpha})$*, and assuming supercompacts this is optimal.*

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The possible LST numbers of these quantifiers are precisely those cardinals that the quantifiers can't identify which are $\leq$ the first supercompact.

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$\beta := \mathrm{cp}(\pi) > \mu$ is a limit cardinal of $\mathcal{B}$, because $\mathcal{A}$ and $\mathcal{B}$ calculate cardinals correctly. Moreover, β is inaccessible in $\mathcal{B}$.
So $\pi(\beta) > \mu$ is inaccessible in $\mathcal{A}$, and must be κ. □

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Theorem (H.T., Osinski)

Suppose that GCH *holds, κ is supercompact and the largest inaccessible, $\alpha \neq 0$ is below the first hyperinaccessible and $0 \neq \beta < \kappa$. There is a generic extension in which* $\mathrm{LST}(I, Q^\alpha)$ *is the β'th element of* Reg_α.

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If $\alpha = 1$ then we also get that $\mathsf{LST}(I) = \mathsf{LST}(I, Q^1)$.

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- Collapses cardinals so κ becomes the next element of Reg_α above μ; and
- Forces $\mathrm{LST}(I, Q^\alpha) \leq \kappa$.

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We force a failure of SCH at some $\lambda \in (\mu, \kappa)$.
We can do this while preserving supercompactness of κ.

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If C is this club, define

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$\mathrm{NM}_\kappa \star \mathrm{Col}_\kappa$ preserves the failure of SCH at λ and makes κ the β'th element of Reg_α.

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j^* exists for a given H iff $j''G \subset H$.
We want to construct a forcing $\mathbb{P}$ which includes $\mathrm{NM}_\kappa \star \mathrm{Col}_\kappa$ such that if G is $\mathbb{P}$ generic, then for all j elementary, $j''G$ is contained in a $j(\mathbb{P})$ generic filter.

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- Even if C were added to M, κ is regular so $C \notin \mathsf{NM}_{j(\kappa)}$;

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- Even if C were added to M, κ is regular so $C \notin \mathrm{NM}_{j(\kappa)}$;
- Even if κ were singularised, Col_κ would be changed by doing this.

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Define a new forcing $\mathbb{Q}_\kappa$ which can modify M to solve these issues:

- Adds an NM_κ club;
- Singularises κ; and
- Does "magic" to make Col_κ work.

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$j(\mathbb{P})$ does not contain any nontrivial $\mathbb{Q}_\gamma$ for $\kappa < \gamma < \nu$.
So $\mathsf{LST}(I, Q^\alpha) \leq \kappa$.

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Thank you for listening!