Strong Measure Zero Sets in the Higher Cantor Space

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Strong Measure Zero Sets

Definition (Borel 1919)

A set $X \subseteq 2^\omega$ is called *strong measure zero* iff for all $f \in \omega^\omega$ there exists a sequence $\langle \eta_n : n \in \omega \rangle$ such that $\forall n : \eta_n \in 2^{f(n)}$ and $X \subseteq \bigcup_{n\in\omega} [\eta_n]$.

Observation

The set $\mathcal{SN}$ of strong measure zero sets is a countably closed ideal which is contained in the ideal of Lebesgue measure zero sets and contains no perfect sets.

Strong Measure Zero Sets

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Borel Conjecture (BC)

Every strong measure zero set is countable.

Consistency of BC

- CON(¬BC) was shown by Sierpiński (Sierpiński 1928) - any Luzin set is strong measure zero
- In fact, $\mathfrak{b} = \aleph_1$ implies ¬BC (Goldstern, Judah, and Shelah 1993)

Theorem (Laver 1976)

BC holds in the Laver model (countable support iteration of Laver forcing of length ω_2).

Strong Measure Zero Sets, Again

Blanket assumption: $\kappa^{<\kappa} = \kappa$.

Definition (Halko 1996)

A set $X \subseteq 2^\kappa$ is called *strong measure zero* iff for all $f \in \kappa^\kappa$ there exists a sequence $\langle \eta_i : i < \kappa \rangle$ such that $\forall i : \eta_i \in 2^{f(i)}$ and $X \subseteq \bigcup_{i<\kappa} [\eta_i]$.

Observation

The set $\mathcal{SN}$ of strong measure zero sets is a $<\kappa^+$-closed ideal containing all singletons.

Strong Measure Zero Sets, Again

Blanket assumption: $\kappa^{<\kappa} = \kappa$.

Definition (Halko 1996)

A set $X \subseteq 2^\kappa$ is called *strong measure zero* iff for all $f \in \kappa^\kappa$ there exists a sequence $\langle \eta_i : i < \kappa \rangle$ such that $\forall i : \eta_i \in 2^{f(i)}$ and $X \subseteq \bigcup_{i<\kappa} [\eta_i]$.

BC(κ)

Every strong measure zero subset of 2^κ has size κ.

Consistency of BC(κ)

- We again have that $\mathfrak{d}_\kappa = \kappa^+$ implies $\neg$BC(κ)(Halko and Shelah 2001)

What about CON(BC(κ))?

Consistency of BC(κ)

Theorem (Halko and Shelah 2001)

Assume $\kappa = \mu^+$. Then the following are equivalent:

- *$A \subseteq 2^\kappa$ is strong measure zero*
- *For every $X \in [\kappa]^\kappa$, the family $(\bar{F} \restriction X)''A$ is not dominating in κ^X, where $(\bar{F} \restriction X)(a) := \langle F_i(a \restriction i) : i \in X\rangle$ and $F_i : 2^i \to |2^i|$ are bijections*

Corollary (Halko and Shelah 2001)

$ZFC \vdash \neg BC(\kappa)$ for successor κ.

Proof.

By the above theorem we have $[2^\kappa]^{<\mathfrak{d}_\kappa} \subseteq \mathcal{SN}$. For $\mathfrak{d}_\kappa = \kappa^+$ we already know that $\neg$BC(κ) follows. □

Consistency of BC(κ)

Theorem (Halko and Shelah 2001)

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Note that for κ inaccessible the theorem fails for 2^κ.

Open Problem

Is BC(κ) consistent?

What About a Laver Iteration?

Definition

A $<\kappa$-closed tree $T \subseteq \kappa^{<\kappa}$ is a κ-Laver tree iff $|\mathrm{succ}_T(\eta)| = \kappa$ for every node $\eta \in T$ above the stem.

Theorem (Khomskii, Koelbing, Laguzzi, and Wohofsky 2023)

Let $\mathbb{P}$ be a forcing notion consisting of κ-Laver trees which is closed under the operation $T \mapsto T \restriction \eta := \{\nu \in T : \nu \subseteq \eta \vee \eta \subseteq \nu\}$. Then $\mathbb{P}$ adds a Cohen κ-real.

What About a Laver Iteration?

Observation (κ inaccessible)

A κ^{++}-c.c. forcing iteration of length κ^{++} that cofinally adds Cohen κ-reals will make any set appearing in an intermediate model strong measure zero.

What About a Laver Iteration?

Open Problem

Is there a $<\kappa$-closed (or even $<\kappa$-distributive) forcing notion adding a dominating κ-real but no Cohen κ-real?

The results of Khomskii, Koelbing, Laguzzi, and Wohofsky 2023 already exclude many natural candidates!

A Related Problem

On ω, Corazza (Corazza 1989) was able to force

$$\mathfrak{c} = \aleph_2 \wedge \mathcal{SN} = [2^\omega]^{<\mathfrak{c}}$$

by using a result of Miller (Miller 1983) to construct a model of "Every set of reals of size continuum can be mapped uniformly continuously onto $[0,1]$".
We follow Corazza's approach.

The Forcing

Definition

Let the forcing PT_f for $f \in \kappa^\kappa$ consist of all $<\kappa$-closed trees $p \subseteq \kappa^{<\kappa}$ such that

1. $\forall \eta \in p \, \forall i \in \mathrm{dom}(\eta): \ \eta(i) < f(i)$
2. $\forall \eta \in p$: $|\mathrm{succ}_p(\eta)| = 1 \vee \mathrm{succ}_p(\eta) = \{\eta^\frown j : j < f(\mathrm{dom}\,\eta)\}$
3. Whenever $b \in [p]$ is a (cofinal) branch of p, then

$$\{i : b \restriction i \text{ is a splitting node of } p\}$$

is a club in κ.

The Forcing

Definition

Let $(\mathcal{P}, \leq_{\mathcal{P}})$ be a forcing notion and $(\leq_i)_{i<\kappa}$ be a sequence of reflexive and transitive binary relations on $\mathcal{P}$ such that $(\leq_i) \subseteq (\leq_j) \subseteq (\leq_{\mathcal{P}})$ for $j < i$. Then

1. $(p_j)_{j<\delta}$ is a *fusion sequence* of length $\delta \leq \kappa$ iff $\forall j < k < \delta : p_k \leq_j p_j$.
2. $\mathcal{P}$ has *Property B* iff
   - $(\mathcal{P}, \leq_{\mathcal{P}})$ is $<\kappa$-closed.
   - Whenever $(p_j)_{j<\delta}$ is a fusion sequence in $\mathcal{P}$, then there exists a *fusion limit* q with $\forall j < \delta : q \leq_j p_j$.
   - If A is a maximal antichain, $p \in \mathcal{P}$ and $i < \kappa$, then there exists a $q \leq_i p$ such that $A \restriction q := \{r \in A \colon r \parallel q\}$ has size $<\kappa$.

The Forcing

Lemma

For every $f \in \kappa^\kappa$ the forcing PT_f is $<\kappa$-closed and satisfies Property B with

$$q \leq_{PT_f} p \iff \forall j \leq i : \mathrm{split}_j(p) = \mathrm{split}_j(q).$$

In particular, it is κ^κ-bounding.

The Model

Assume $V \models GCH$ and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ be a $\leq\kappa$-supported forcing iteration with

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = PT_{f_\alpha}$$

where each $f \in \kappa^\kappa \cap V$ appears cofinally. For technical reasons we require $f_\alpha \equiv 2$ for $\alpha = 0$ or $\operatorname{cf}\alpha = \kappa^+$.

Theorem

$\mathbb{P}_\alpha$ is $<\kappa$-closed, κ^{++}-c.c. and satisfies an iteration version of Property B.

Easy Inclusion

Theorem

$V^{\mathbb{P}} \models [2^\kappa]^{\leq\kappa^+} \subseteq \mathcal{SN}$.

Proof Sketch.

In the extension, let $X \in [2^\kappa]^{\leq\kappa^+}$ and $f \in \kappa^\kappa$. By the κ^{++}-c.c., they both already appear at a stage $\alpha < \kappa^{++}$. Find an $h \in \kappa^\kappa \cap V$ with $f \leq h$ and a $\beta > \alpha$ such that $f_\beta(i) = |2^{h(i)}|$. By a density argument, the β-th generic real will encode an h-cover of X. □

$\mathcal{SN} \subseteq [2^\kappa]^{\leq \kappa^+}$

Lemma (Key Lemma)

Let $p \in \mathbb{P}$ force $\dot{\tau} \in 2^\kappa$ and $\dot{\tau} \notin V$. Then there exists a $q \leq p$ and a uniformly continuous function $f : 2^\kappa \to [q(0)]$ in V such that

$$q \Vdash f(\dot{\tau}) = \dot{s}_0,$$

where $\dot{s}_0$ denotes the first Sacks real (recall that $f_0 \equiv 2$).

Lemma

Let $p \in \mathbb{P}$ be a condition. Then there exists a uniformly continuous $g : [p(0)] \to 2^\kappa$ and for each $x \in 2^\kappa \cap V$ a condition $q_x \leq p$ such that

$$q_x \Vdash \check{x} = g(\dot{s}_0).$$

Theorem

In $V^{\mathbb{P}}$, every set $X \in [2^\kappa]^{\kappa^{++}}$ can be uniformly continuously mapped onto 2^κ.

Being strong measure zero is preserved by uniformly continuous functions, therefore we are done.

Stationary Strong Measure Zero

Observation

$X \subseteq 2^\kappa$ is strong measure zero iff for all $f \in \kappa^\kappa$ there exists an $(\eta_i)_{i<\kappa}$ such that $\forall i < \kappa: \ \eta_i \in 2^{f(i)}$ and for each $x \in X$

$$|\{i < \kappa : x \in [\eta_i]\}| = \kappa.$$

Definition (Halko 1996)

$X \subseteq 2^\kappa$ is *stationary* strong measure zero iff for all $f \in \kappa^\kappa$ there exists an $(\eta_i)_{i<\kappa}$ such that $\forall i < \kappa: \ \eta_i \in 2^{f(i)}$ and for each $x \in X$

$$\{i < \kappa : x \in [\eta_i]\}$$

is stationary.

Stationary Strong Measure Zero

Theorem

In the Corazza-type model $V^{\mathbb{P}}$, every strong measure zero set is stationary strong measure zero. However, under $|2^\kappa| = \kappa^+$, there exists a strong measure zero set that is not stationary strong measure zero.

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