The κ^+-Borel hierarchy and changes of topologies

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Generalised Baire Space and Large Cardinals

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From joint works (some still in progress) with Luca Motto Ros, Beatrice Pitton, and/or Philipp Schlicht

FWF

Der Wissenschaftsfonds.

The classical case:

A **Borel space** $(X, \mathscr{B})$ is a set X with a σ-algebra $\mathscr{B}$ on it such that

- $\mathscr{B}$ is countably generated and separates points of X, or, equivalently,
- $\mathscr{B}$ is generated by a metrizable second-countable topology on X.

A **standard Borel space** $(X, \mathscr{B})$ is a Borel space such that, equivalently:

- it is Borel isomorphic to a Borel subset of the Baire space ${}^{\omega}\omega$, or
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The topology is not unique.

Thus, the Borel sets of a Borel space can be stratified in different Borel hierarchies.

Let κ be a cardinal of (uncountable) cofinality μ such that $2^{<\kappa} = \kappa$.

Definition

A κ^+-**Borel space** $(X, \mathscr{B})$ is a set X with a κ^+-algebra $\mathscr{B}$ on it such that $\mathscr{B}$ is generated by a family of size $\leq \kappa$ and separates points of X.

Definition

A κ^+-Borel space $(X, \mathscr{B})$ is **standard** if it is κ^+-Borel isomorphic to a κ^+-Borel subset of the generalized Baire space ${}^{\mu}\kappa$.

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What "Polish" topology could generate this?

Many candidates of "Polish-like" topologies:

- $\mathbb{G}$-Polish spaces,
- Sph. complete $\mathbb{G}$-Polish spaces,
- $\mathbf{SC}_\kappa$-spaces with a metrizable-like condition,
- $\mathbf{fSC}_\kappa$-spaces with a metrizable-like condition,
- ...

Theorem (A., Motto Ros, Schlicht/A., Motto Ros)

All above classes coincide up to κ^+-Borel isomorphism.

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Theorem (A., Motto Ros, Schlicht/A., Motto Ros)

The following are equivalent for a κ^+-Borel space $(X, \mathcal{B})$:

- *$(X, \mathcal{B})$ is standard κ^+-Borel,*
- *$\mathcal{B}$ is generated by a $\mathbf{fSC}_\kappa$ topology,*
- *$\mathcal{B}$ is generated by a sph. complete $\mathbb{G}$-Polish topology.*

Metrizable-like topologies

Many candidates of "metrizable-like" topologies:

- μ-metrizable spaces,
- NS_μ^δ-spaces for some $\delta \leq \mu$,
- μ-uniformly based (μ, μ)-paracompact spaces,
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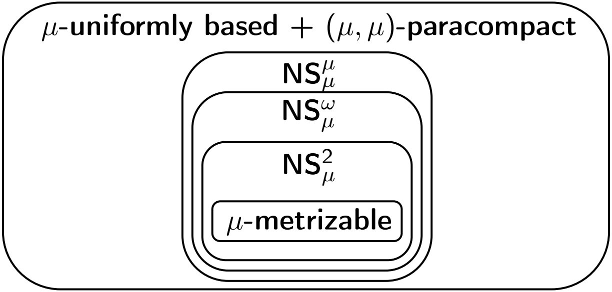

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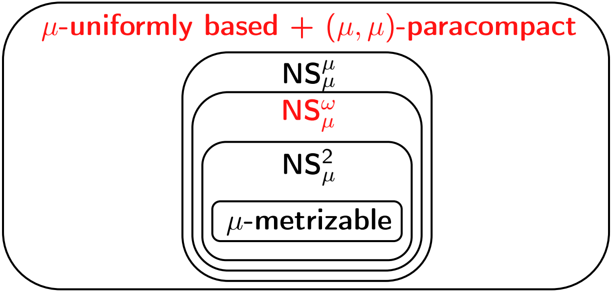

A family $\mathcal{A}$ of subsets of X is locally $<\gamma$-sized if every point of X has an open neighborhood that intersect $<\gamma$-many elements of $\mathcal{A}$.

A family of sets is called $\mathsf{NS}_\delta^\gamma$ if it is the union of δ-many locally $<\gamma$-sized families.

We call X a $\mathsf{NS}_\delta^\gamma$-**space** if it is (regular Hausdorff and) it has a base for the topology that is a $\mathsf{NS}_\delta^\gamma$-family.

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Theorem (Bing-Nagata-Smirnov Metrization Theorem)

The following are equivalent:

1. *X is metrizable*
2. *X is a $\mathsf{NS}^{\omega}_{\omega}$-space.*
3. *X is a NS^{2}_{ω}-space.*

Remark: weight $\leq \delta$ implies NS^{2}_{δ}-space.

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The μ-**uniform local base game**: at every round $\alpha < \mu$, player I pick a point $x_\alpha \in X$, and player II replies with an open set V_α containing x_α.

I	x_0		x_1		...	x_γ		...
II		V_0		V_1	...		V_γ	...

At the end of the game, player II wins if $\bigcap_{\alpha<\mu} V_\alpha = \varnothing$ or if $\{V_\alpha \mid \alpha < \mu\}$ is a local base of a point $x \in X$, otherwise I wins.

X is μ-**uniformly based** if player II has a winning strategy.

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Theorem (A., Motto Ros)

X is metrizable if and only if it is $((\omega,\omega)$-$)$paracompact and ω-uniformly based.

Theorem (Folklore?)

Let X be a set and let $\mathcal{B}$ be a κ^+-algebra on X. The following are equivalent:

- $(X, \mathscr{B})$ *is a κ^+-Borel space,*
- $\mathscr{B}$ *is generated by a (regular Hausdorff) topology of weight $\leq \kappa$,*
- $\mathscr{B}$ *is generated by a metrizable-like topology of weight $\leq \kappa$,*
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What does the κ^+-Borel hierarchy generated by these topologies look like?

All topological spaces are assumed to be Hausdorff and regular.

Let (X, τ) be a topological space and γ be a cardinal.

Definition

The γ-hierarchy is defined by recursion on α:

- $\mathbf{\Sigma}^0_1(X, \tau, \gamma) = \tau$;
- $\mathbf{\Sigma}^0_\alpha(X, \tau, \gamma) = \{\bigcup \mathcal{A} \mid \mathcal{A} \subseteq \bigcup_{1 \leq \beta < \alpha} \mathbf{\Pi}^0_\beta(X, \tau, \gamma), |\mathcal{A}| < \gamma\}$;
- $\mathbf{\Pi}^0_\alpha(X, \tau, \gamma) = \{X \setminus A \mid A \in \mathbf{\Sigma}^0_\alpha(X, \tau, \gamma)\}$.

We also set $\mathbf{\Delta}^0_\alpha(X, \tau, \gamma) = \mathbf{\Sigma}^0_\alpha(X, \tau, \gamma) \cap \mathbf{\Pi}^0_\alpha(X, \tau, \gamma)$.

We denote by $\mathrm{Bor}(X, \tau, \gamma)$ the smallest γ-algebra of sets generated by τ.

Definition

We say that the γ-hierarchy is **increasing (above δ)** if for all $\alpha < \beta$ (resp., such that $\delta \leq \alpha$) we have

$$\mathbf{\Sigma}^0_\alpha(X, \tau, \gamma) \cup \mathbf{\Pi}^0_\alpha(X, \tau, \gamma) \subseteq \mathbf{\Delta}^0_\beta(X, \tau, \gamma).$$

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Remark: by definition,

$$\mathbf{\Sigma}^0_1(X,\tau,\gamma) \subseteq \mathbf{\Pi}^0_2(X,\tau,\gamma) \subseteq \mathbf{\Sigma}^0_3(X,\tau,\gamma),$$

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Thus, the γ-hierarchy is always increasing above 2, and it is increasing if and only if $\mathbf{\Sigma}^0_1(X,\tau,\gamma) \subseteq \mathbf{\Sigma}^0_2(X,\tau,\gamma)$.

The κ^+-hierarchy allows to stratify κ^+-Borel sets in classes.

Remark: $\mathrm{Bor}(X, \tau, \kappa^+) = \bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}^0_\alpha(X, \tau, \kappa^+) = \bigcup_{\alpha<\kappa^+} \mathbf{\Pi}^0_\alpha(X, \tau, \kappa^+)$.

Remark: If X is regular Hausdorff of weight $\leq \kappa$, then $\tau \subseteq \mathbf{\Sigma}^0_2(X, \tau, \kappa^+)$. Thus the κ^+-hierarchy is (always) increasing.

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In general, having weight $\leq \kappa$ does not ensure $\tau \subseteq \mathbf{\Sigma}^0_2(X, \tau, \kappa)$.
This is true however for NS^ω_μ-spaces.

Question

Is the κ-hierarchy of any μ-uniformly based (μ, μ)-paracompact space increasing?

Thus, when κ is singular, we have two hierarchies (the κ^+-hierarchy and the κ-hierarchy), both of length κ^+, stratifying κ^+-Borel sets.

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Theorem (A., Motto Ros, Pitton)

Let (X, τ) be a space (of weight $\leq \kappa$) and $\alpha < \kappa^+$ be an infinite ordinal.

1. *If α is even, then*

$$\boldsymbol{\Sigma}^0_{1+\alpha}(X, \tau, \kappa) = \boldsymbol{\Sigma}^0_{1+\frac{\alpha}{2}}(X, \tau, \kappa^+),$$

$$\boldsymbol{\Pi}^0_{1+\alpha}(X, \tau, \kappa) = \boldsymbol{\Pi}^0_{1+\frac{\alpha}{2}}(X, \tau, \kappa^+).$$

2. *If α is odd, then*

$$\boldsymbol{\Sigma}^0_{1+\alpha}(X, \tau, \kappa) = \boldsymbol{\Pi}^0_{1+\alpha}(X, \tau, \kappa) = \boldsymbol{\Delta}^0_{1+\alpha}(X, \tau, \kappa).$$

Furthermore, if (X, τ) is a NS^ω_μ-space, the same is true for finite ordinals.

We call a topological space μ-**additive** if every intersection of $< \mu$-many open sets is still open.

Let (X, τ) be a topological space. Then there is always a smallest μ-additive topology τ' refining τ. Notice that

$$\tilde{\mathcal{B}} = \{\bigcap \mathcal{A} \mid \mathcal{A} \subseteq \tau, |\mathcal{A}| < \mu, \bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} \mathbf{cl}(A)\}$$

is a family of τ-closed set generating τ'. Thus, $\tau' \subseteq \mathbf{\Sigma}^0_2(X, \tau, \kappa^+)$.

Corollary

Let (X, τ) be a space and let τ' be the μ-additive refinement of τ. Then, for every infinite ordinal $\alpha < \kappa^+$,

$$\mathbf{\Sigma}^0_\alpha(X, \tau, \kappa^+) = \mathbf{\Sigma}^0_\alpha(X, \tau', \kappa^+)$$

In general, if (X, τ) is a paracompact space and τ' is the μ-additive refinement of τ, then (X, τ') need not be paracompact.

Theorem (A., Motto Ros)

Let (X, τ) be a μ-uniformly based, (μ, μ)-paracompact space. Let τ' be the μ-additive refinement of τ. Then (X, τ') is a paracompact.

Corollary

Let (X, τ) be a μ-uniformly based, (μ, μ)-paracompact space. Let τ' be the μ-additive refinement of τ. Then (X, τ') is (a NS_μ^ω-space) homeomorphic to a subset of ${}^\mu\kappa$.

Thus, it requires to change at most the first ω-many levels of the κ-hierarchy to make it "nice".

Let either $\gamma = \kappa$ or $\gamma = \kappa^+$.

Definition

We say that the γ-hierarchy is **collapsing** if for some $\alpha < \kappa^+$,

$$\mathbf{\Sigma}^0_\alpha(X, \tau, \gamma) = \mathbf{\Pi}^0_\alpha(X, \tau, \gamma) = \mathbf{\Delta}^0_\alpha(X, \tau, \gamma) = \mathbf{Bor}(X, \tau, \gamma).$$

Remark: for a space of weight $\leq \kappa$, the κ-hierarchy collapses if and only if the κ^+-hierarchy collapses.

Proposition (A., Motto Ros, Pitton)

Let $(X, \mathcal{B})$ be a κ^+-Borel space. Then for all topologies τ, τ' of weight $\leq \kappa$ generating $\mathcal{B}$ we have that the κ^+-hierarchy of (X, τ) collapses if and only if the κ^+-hierarchy of (X, τ') collapses.

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Proof: if τ, τ' are two topologies of weight $\leq \kappa$ generating the same κ^+-Borel structure, then there is $\alpha < \kappa^+$ such that for all $\beta > \alpha$

$$\mathbf{\Sigma}^0_\beta(X, \tau, \kappa^+) = \mathbf{\Sigma}^0_\beta(X, \tau', \kappa^+).$$

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$$\mathbf{\Sigma}^0_\beta(X, \tau, \kappa^+) = \mathbf{\Sigma}^0_\beta(X, \tau', \kappa^+).$$

Corollary

Let (X, τ) be a space of weight $\leq \kappa$ with a κ^+-Borel embedding $f : {}^\kappa 2 \to X$. Then, the κ^+-hierarchy on (X, τ) does not collapse.

C. Agostini and L. Motto Ros.
Generalized Polish spaces for all cardinals.
Manuscript in preparation.

C. Agostini, L. Motto Ros, and B. Pitton.
Generalized borel sets.
Manuscript in preparation.

C. Agostini, L. Motto Ros, and P. Schlicht.
Generalized Polish spaces at regular uncountable cardinals.
Accepted on Journal of London Mathematical Society.

A. Andretta and L. Motto Ros.
Souslin quasi-orders and bi-embeddability of uncountable structures.
Mem. Amer. Math. Soc., 277(1365):vii+189, 2022.

A. Kechris.
Classical descriptive set theory, volume 156.
Springer Science & Business Media, 2012.