OPEN QUESTIONS ON GENERALISED BAIRE SPACES

Abstract. Open questions collected at the seventh workshop on generalised Baire spaces that took place at the University of Bristol in February 2024.

See https://www.bristol.ac.uk/maths/events/2024/philip-welch-event-.html.

Please add (some) open questions from your talk (and additional problems) below and edit any time (participants of the workshop). Please copy in your changes all at the same time to avoid conflicts. If you send an email (see end of the file) after you've made edits, I'll update the file on the webpage.

1. Borel reducibility and model theory

Problem 1.1. (Miguel) Do all analytic strong measure 0 subsets of ${}^{\kappa}2$ have size $\leq\kappa$?

Problem 1.2. (Miguel) Do all analytic strong measure 0 subsets of ${}^{\kappa}2$ have size $\leq\kappa$?

Problem 1.3. (Miguel) Is it consistent that there is a stable unsuperstable theory T such that $\cong_T$ is Δ^1_1?

Problem 1.4. (Miguel) Are filter reflection and $=^{\kappa}_{X}\hookrightarrow_{B}=^{2}_{Y}$ equivalent?

Problem 1.5. (Miguel) For which values of θ and $S\subseteq\kappa$ stationary, does $=^{\theta}_{X}\hookrightarrow=^{2}_{S}$ hold?

Recall that $=^{\theta}_{S}\subseteq\theta^{\kappa}\times\theta^{\kappa}$ *for* $2\leq\theta\leq\kappa$ *is defined as* $\eta=^{\theta}_{S}\xi$ *if* $\{\alpha<\kappa\mid\eta(\alpha)\neq\xi(\alpha)\}\cap S$ *is not stationary.*

Problem 1.6. (Miguel) Can the cardinal arithmetic assumptions of the Borel reducibility main gap be relaxed?

Problem 1.7. (Miguel) Is isomorphism of graphs of size κ Σ^1_1-complete?

Problem 1.8. (Miguel) Is it consistent that the definable sets in $\mathcal{M}_{\kappa^+,\kappa}$ are precisely the Borel* sets under isomorphism?

2. Dichotomies

Problem 2.1. (Dorottya) Does the κ-perfect set property for κ-analytic sets imply the open graph dichotomy for κ-analytic sets (equivalently for closed sets)?

Problem 2.2. (Dorottya) Does the PSP for closed sets already imply the statements in problem 2.1?

Problem 2.3. (Dorottya) Do any of the previous statements in problems 2.1 and 2.2 imply $\mathrm{CCP}_\kappa(\Sigma^1_1,\mathsf{D}_\alpha)$, i.e. the version for definable families of closed sets.

Recall that $\mathrm{CCP}_\kappa(X)$ *states that for any* κ*-ideal* $\mathcal{I}$ *on* X *generated by a family of closed sets, either* $X\in\mathcal{I}$ *or there exists a continuous function* $f\colon{}^{\kappa}\kappa\to X$ *such that* $f(N_t)\in\mathcal{I}^+$ *for all* $t\in{}^{<\kappa}\kappa$*.* $\mathrm{CCP}_\kappa(\mathsf{D}_\kappa)$ *states that* $\mathrm{CCP}_\kappa(X)$ *holds for all subsets* X *of* ${}^{\kappa}\kappa$ *definable from a* κ*-sequence of ordinals.*

Problem 2.4. (Dorottya) Does $\mathrm{CCP}_\kappa(\mathsf{D}_\kappa)$ have at least the consistency strength of a Mahlo cardinal?

Problem 2.5. (Philipp S.) Is the consistency strength of the Hurewicz dichotomy for Π^1_1 subsets of ${}^{\kappa}2$ an inaccessible cardinal?

(Stern published a flawed proof that one does not need an inaccessible for ${}^{\omega}2$*.)*

3. Uniformisation

Problem 3.1. (Philipp S.) Does the version of the Lusin-Novikov uniformisation theorem for ${}^{\kappa}\kappa$ always fail, i.e., can one prove that exists a κ-Borel relation with sections of size $\leq\kappa$ that does not admit a κ-Borel measurable uniformisation?

4. Kurepa trees

Problem 4.1. (Claudio) Is there a κ-Kurepa subtree T of ${}^{<\kappa}\kappa$ such that the κ^+-Borel hierarchy for the space $[T]$ collapses? (This means there exists some $\alpha<\kappa^+$ such that every κ-Borel set is $\mathbf{\Sigma}^0_\alpha$.)

5. Forcing

Problem 5.1. (Yurii) Is there a $<\kappa$-closed or $<\kappa$-distributive forcing adding a dominating κ-real without adding κ-Cohen reals?

6. Combinatorics

Problem 6.1. (Nick) Is the Borel conjecture consistent for κ?

Problem 6.2. (Miguel) Is the an analytic κ-MAD family in ${}^{\kappa}2$?

7. Löwenheim-Skolem numbers

Problem 7.1. (Christopher) Is it consistent that the LST number of the Härtig quantifier I is singular?

Recall that $\mathcal{A} \vDash Ix, y(\varphi(x), \psi(y))$ *holds if the sets defined by* φ *and* ψ *have the same cardinality.* $\mathrm{LST}(I)$ *is the least cardinal* κ *such that any* $\mathcal{L}$ *structure* $\mathcal{A}$ *has an* $\mathcal{L} \cup \{I\}$ *elementary substructure of size less than* κ.

References

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